AP, GP Problem Set

1. **[ACJC Prelims 17]**
   Abbie and Benny each take a $50 000 study loan for their 3-year undergraduate program, disbursed on the first day of the program. The terms of the loan are such that during the 3- year period of their studies, interest is charged at 0.1% of the outstanding amount at the end of each month. Upon graduation, interest is charged at 0.375% of the outstanding amount at the end of each month.

   (a) Since the interest rate is lower during her studies, Abbie decides that she will make a constant payment at the beginning of each month from the start of the program for its entire duration.

   i. Find the amount, correct to the nearest cent, Abbie needs to pay at the beginning of each month so that the outstanding amount after interest is charged remains at $50 000 at the end of every month. [2]

   ii. After graduating, Abbie intends to increase her payment to a constant $\$k$ at the beginning of every month. Show that the outstanding amount Abbie owes the bank at the end of n months after graduation, and after interest is charged, is

   $$\$\left[1.00375^n(50000) - \frac{803}{3}k(1.00375^n - 1)\right].$$

   [2]

   (b) Benny wishes to begin his loan repayment only after graduation. Like Abbie, he aims to repay the loan at the end of exactly 10 years after graduation. Leaving your answer to the nearest cent, find

   i. the constant amount Benny needs to pay each moth in order to do this, [3]

   ii. the amount of interest Benny pays altogether. [2]

2. **[AJC Prelims 17]**
   The sum to n terms of a series is given by $S_n = n\ln 2 - \dfrac{n^2-1}{\mathrm{e}}$.

   Find an expression for the nth term of the series in terms of n.

   Show that the terms of the series follow an arithmetic progression. [4]

3. [**AJC Prelims 17**]
There are 25 toll stations, represented by $T_1, T_2, T_3, \ldots, T_{25}$ along a 2000 km stretch of highway. T_1 is located at the start of the highway and T_2 is located x km from T_1. Subsequently, the distance between two consecutive toll stations is 2 km more than the previous distance.

Find the range of values x can take. [3]

Use $x = 60$ for the rest of this question.

Each toll station charges a fee based on the distance travelled from the previous toll station. The fee structure at each toll station is as follows: For the first 60 km, the fee per km will be 5 cents. For every additional 2 km, the fee per km will be 2% less than the previous fee per km.

(a) Find, in terms of n, the amout of fees a driver will need to pay at T_n. [3]

(b) Find the total amount of fees a driver will need to pay, if he drives from T_1 to T_n. Leave your answer in terms of n. [4]

More toll station are built along the highway in the same manner, represented by $T_{26}, T_{27}, T_{28}, \ldots$ beyond the 2000 km stretch.

(c) If a driver starts driving from T_1 and only has \$200, at which toll station will he not have sufficient money for the fees? [2]

4. [**CJC Prelims 17**]
Kumar wishes to purchase a gift priced at \$280 for his mother.

Starting from January 2017,

- Kumar saves \$100 in his piggy bank on the 1st day of each month;
- Kumar donates 30% of his money in his piggy bank to charity on the 15th day of each month and
- Kumar's father puts an additional \$20 in Kumar's piggy bank on the 25th day of each month.

(a) Find the amount of money in Kumar's piggy bank at the end of March 2017. [2]

(b) Show that the amount in Kumar's piggy bank at the end of n months is $300(1 - 0.7^n)$. [3]

(c) At the end of which month will Kumar first be able to purchase the gift for his mother? [2]

5. **[DHS Prelims 17]**
A geometric sequence $T_1, T_2, T_3, \ldots$ has a common ratio of e. Another sequence $U_1, U_2, U_3, \ldots$ is sch that $U_1 = 1$ and

$$U_r = \ln T_r - 3 \quad \text{for all } r \geq 1.$$

(a) Prove that the sequence $U_1, U_2, U_3, \ldots$ is arithmetic. [2]

A third sequence $W_1, W_2, W_3, \ldots$ is such that $W_1 = \frac{1}{2}$ and

$$W_{r+1} = W_r + U_r \quad \text{for all } r \geq 1.$$

(a) By considering $\sum_{r=1}^{n-1}(W_{r+1} - W_r)$, show that

$$W_n = \frac{1}{2}(n^2 - n + 1).$$

[3]

6. **[TPJC Prelims 17]**
Timber cladding is the application of timber planks over timber planks to provide the layer intended to control the infiltration of weather elements.

(a) Using method A, 20 rectangular planks are used and the lengths of the planks form an arithmetic progression with common difference d cm. The shortest plank has length 65 cm and the longest plank has length 350 cm.

i. Find the value of d. [2]

ii. Find the total length of all the planks. [2]

(b) Using method B, a long plank of 2000 cm is sawn off by a machine into n smaller rectangular planks. The length of the first plank is a cm and each successive plank is $\frac{8}{9}$ as long as the preceding plank.

i. Show that the total length of the planks sawn off can never be greater than k times the length of the first plank, where k is an integer to be determined. [2]

ii. Given that $a = 423$, find the greatest possible integral value of n and the corresponding length of the shortest plank. [4]

7. An arithmetic series has first term a and common difference d, where a and d are non-zero. The first, sixth and tenth terms form the first three terms of a geometric series with common ratio r.

   (a) Show that $5r^2 - 9r + 4 = 0$. [3]

   (b) Determine, with reason, if the geometric series is convergent. If it is, find, in terms of a, the sum to infinity. [4]

   (c) The sum of the first n terms of the arithmetic series is denoted by S_n Given that $a > 0$, find the set of possible values of n for which S_n is at least $12a$. [4]

Answers

1. (a) i. \$49.95.
       ii. No. \$ 516.26 per month.
   (c) i. \$535.17 per month.
       ii. \$14220.43.

2. $\ln 2 - \frac{1}{\mathrm{e}}(2n-1)$.

3. $0 < x \leq \frac{181}{3}$.

   (a) $7.9 - 4.9(0.98^{n-2})$.
   (b) $7.9n + 245(0.98^{n-1}) - 252.9$.
   (c) 45th.

4. (a) \$197.10.
   (c) August 2017.

6. (a) i. $d = 15$.
       ii. $S_{20} = 4150$ cm.
   (b) i. $k = 9$.
       ii. $n = 6$. Length $= 235$ cm.

7. (b) It converges since $-1 < r = \frac{4}{5} < 1$.
   $S_\infty = 5a$.
   (c) $\{n \in \mathbb{Z} : 19 \leq n \leq 32\}$.