Maclaurin Series Problem Set

1. **[ACJC Prelims 17]**

   (a) Expand $(k+x)^n$, in ascending powers of x, up to and including the term in x^2, where k is a non-zero real constant and n is a negative integer. [3]

   (b) State the range of values of x for which the expansion is valid. [1]

   (c) In the expansion of $(k+y+3y^2)^{-3}$, the coefficient of y^2 is 2. By using the expansion in (a), find the value of k. [3]

2. **[ACJC Prelims 17]**
   It is given that $\mathrm{e}^y = (1+\sin x)^2$.

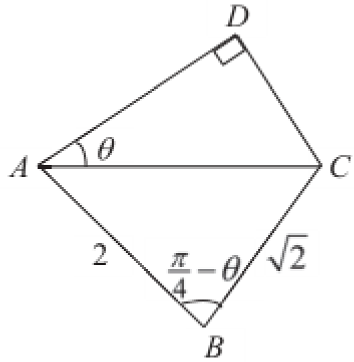
   (a) Show that

   $$\mathrm{e}^y\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right) = 2(\cos 2x - \sin x).$$

   By repeated differentiation, find the series expansion of y in ascending powers of x, up to and including the term in x^3, simplifying your answer. [5]

   (b) Show how you can use the standard series expansion(s) to verify that the terms up to x^3 for your series expansion of y in (a) are correct. [3]

3. **[AJC Prelims 17]**
The diagram shows a quadrilateral $ABCD$, where $AB = 2$, $BC = \sqrt{2}$, $\angle ABC = \frac{\pi}{4} - \theta$ radians and $\angle CAD = \theta$ radians.



Show that $AC = \sqrt{6 - 4\cos\theta - 4\sin\theta}$. [2]

Given that θ is small enough for θ^3 and higher powers of θ to be neglected, show that

$$AD \approx a + b\theta + c\theta^2,$$

where a, b and c are constants to be determined. [5]

4. **[AJC Prelims 17]**
A curve C has equation $y = f(x)$. The equation of the tangent to the curve C at the point where $x = 0$ is given by $2x - ay = 3$ where a is a positive constant.

It is also given that $y = f(x)$ satisfies the equation

$$(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and that the third term in the Maclaurin's expansion of $f(x)$ is $\frac{1}{3}x^2$.

Find the value of a. Hence, find the Maclaurin's series for $f(x)$ in ascending powers of x, up to and including the term in x^3. [7]

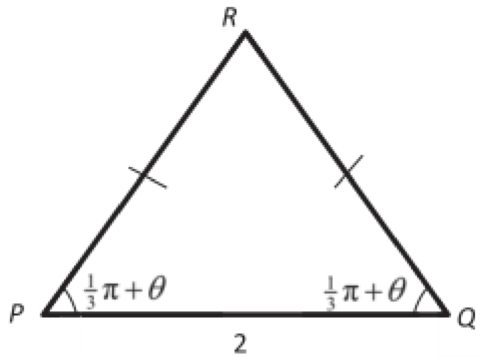
5. **[CJC Prelims 17]**

   (a) Given that the first two terms in the series expansion of $\sqrt{4-x}$ are equal to the first two terms in the series expansion of $p+\ln(q-x)$, find the constants p and q. [5]

   (b) i. Given that $y = \tan^{-1}(ax+1)$ where a is a constant, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = a\cos^2 y$. Use this result to find the Maclaurin series for y in terms of a, up to and including the term in x^3. [5]

   ii. Hence, or otherwise, find the series expansion of $\frac{1}{1+(4x+1)^2}$ up to and including the term in x^2. [3]

6. **[DHS Prelims 17]**
   In the isosceles triangle PQR, $PQ = 2$ and $\angle QPR = \angle PQR = \frac{\pi}{3} + \theta$ radians. The area of triangle PQR is denoted by A.



Given that θ is a sufficiently small angle, show that

$$A = \frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta} \approx a + b\theta + c\theta^2,$$

for constants a, b and c to be determined in exacct form. [5]

7. [**HCI Prelims 17**]

(a) It is given that $\ln y = 2\sin x$. Show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -y\ln y + \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2.$$

[2]

(b) Find the first four terms of the Maclaurin series for y in ascending powers of x. [4]

(c) Using appropriate expansions from MF26, verify the expansion found in part (b). [2]

(d) Given that x is sufficiently small for x^4 and higher powers of x to be neglected, deduce an approximation for $\mathrm{e}^{2\sin x - \ln(\sec x)}$. [2]

8. [**IJC Prelims 17**]

(a) The variables x and y are related by

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} + ky = 2 \quad \text{and} \quad y = 1 \text{ at } x = 0,$$

where k is a constant. Show that

$$(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1+k)\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0.$$

[1]

By further differentiation of this result, find the Maclaurin series for y, up to and including the term in x^3, giving the coefficients in terms of k. [4]

(b) Given that x is small, find the series expansion of

$$g(x) = \frac{1}{\sin^2\left(2x + \frac{\pi}{2}\right)}$$

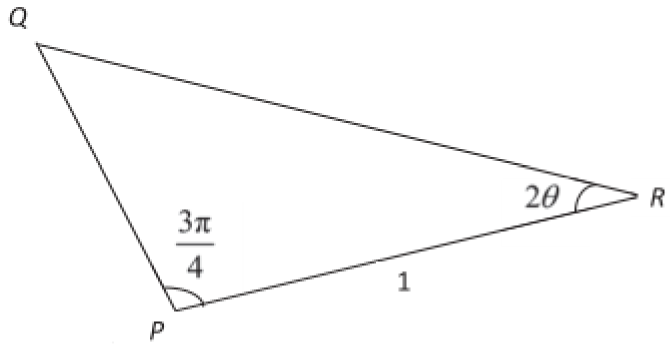
in ascending powers of x, up to and including the term in x^2.

If the coefficient of x^2 in the expansion of $g(x)$ is equal to twice the coefficient of x^2 in the Maclaurin series for y found in part (a), find the value of k. [4]

9. **[TPJC Prelims 17]**

(a) Find the series expansion of $\mathrm{e}^{2x}\ln(1+3x)$, where $-\frac{1}{3} < x \leq \frac{1}{3}$, in ascending powers of x, up to and including the term in x^3. [3]

(b) In the triangle PQR as shown in the diagram below, $PR = 1$, $\angle QPR = \frac{3\pi}{4}$ radians and $\angle PRQ = 2\theta$ radians.



i. Show that $QR = \dfrac{1}{\cos 2\theta - \sin 2\theta}$. [4]

ii. Given that θ is a sufficiently small angle, show that

$$QR \approx 1 + a\theta + b\theta^2,$$

for constants a and b to be determined. [4]

10. **[TJC Prelims 17]**
Given that $\mathrm{e}^y = \sqrt{\mathrm{e} + x + \sin x}$, show that

$$2\mathrm{e}^{2y}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\mathrm{e}^{2y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \sin x = 0.$$

[2]

(a) Find the values of y, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x = 0$. Hence, find in terms of e, the Maclaurin's series for $\ln(\mathrm{e} + x + \sin x)$, up to and including the term in x^2. [4]

(b) By using appropriate standard series expansions from the List of Formulae (MF26), verify the correctness of the first three terms in the series expansion for $\ln(\mathrm{e} + x + \sin x)$ found in part (a). [3]

(c) Use your answer to part (a) to give an approximation for

$$\int_0^{\mathrm{e}^{-1}} \frac{2\mathrm{e} - 4x}{\mathrm{e}^2\ln(\mathrm{e} + x + \sin x)}\,\mathrm{d}x,$$

giving your answer in terms of e. [3]

Answers

1. (a) $k^n(1 + \frac{n}{k}x + \frac{n(n-1)}{2k^2}x^2 + \ldots)$.
   (b) $-|k| < x < |k|$.
   (c) 0.642.

2. $y = 2x - x^2 + \frac{1}{3}x^3 + \ldots$

3. $a = \sqrt{2}, b = -\sqrt{2}, c = -\frac{\sqrt{2}}{2}$.

4. $a = 3, -1 + \frac{2}{3} + \frac{1}{3}x^2 - \frac{5}{27}x^3 + \ldots$

5. (a) $p = 2 - \ln 4$.
   (b) i. $\frac{\pi}{4} + \frac{1}{2}ax - \frac{1}{4}a^2x^2 + \frac{1}{12}a^3x^3 + \ldots$
   ii. $\frac{1}{2} - 2x + 4x^2$.

6. $\sqrt{3} + 4\theta + 4\sqrt{3}\theta^2$.

7. (b) $y = 1 + 2x + 2x^2 + x^3 + \ldots$
   (d) $\mathrm{e}^{2\sin x}\cos x \approx 1 + 2x + \frac{3}{2}x^2 + \ldots$

8. (a) $y = 1 + (2-k)x + (\frac{3k-6}{2})x^2 + (k^2 - 6k + 8)x^3 + \ldots$
   (b) $1 + 4x^2 + \ldots$.
   $k = \frac{10}{3}$.

9. (a) $3x + \frac{3}{2}x^2 + 6x^3 + \ldots$
   (b) $a = 2, b = 6$.

10. (a) $\ln(\mathrm{e} + x + \sin x) = 1 + \frac{2}{\mathrm{e}}x - \frac{2}{\mathrm{e}^2}x^2 + \ldots$
    (b) $\ln(\mathrm{e}^4 + 2\mathrm{e}^2 - 2) - 4$.