Sigma Notation Problem Set

1. **[ACJC Prelims 17]**
   By writing
   $$\sin\left(x+\frac{1}{4}\right)\pi - \sin\left(x-\frac{3}{4}\right)\pi$$
   in terms of a single trigonometric function, find $\displaystyle\sum_{r=1}^{n}\cos\left(x-\frac{1}{4}\right)\pi$,
   leaving your answer in terms of n. [4]

2. **[AJC Prelims 17 (modified)]**

   (a) Use partial fractions to show that
   $$\sum_{n=1}^{N}\frac{1}{4n^2-1} = \frac{1}{2} - \frac{1}{2(2N+1)}$$
   [3]

   (b) Hence find $\displaystyle\sum_{n=1}^{2N}\frac{1}{4(n+1)^2-1}$.

   Deduce that $\displaystyle\sum_{n=1}^{2N}\frac{1}{(2n+3)^2}$ is less than $\dfrac{1}{6}$. [5]

3. **[CJC Prelims 17]**

   (a) Express $\dfrac{r+1}{(r+2)!}$ in the form
   $$\frac{A}{(r+1)!} + \frac{B}{(r+2)!},$$
   where A and B are integers to be found. [2]

   (b) Find $\displaystyle\sum_{r=1}^{n}\frac{r+1}{3(r+2)!}$. [3]

   (c) Hence evaluate $\displaystyle\sum_{r=0}^{\infty}\frac{r+1}{3(r+2)!}$. [2]

4. **[DHS Prelims 17]**

   Given that $\sum_{k=1}^{n} k!(k^2+1) = (n+1)!n$, find

$$\sum_{k=1}^{n-1} (k+1)!(k^2+2k+2).$$

[3]

5. **[HCI Prelims 17]**

   The sum, S_n of the first n terms of a sequence, $u_1, u_2, u_3, \ldots$ is given by

$$S_n = b - \frac{3a}{(n+1)!},$$

   where a and b are constants.

   (a) It is given that $u_1 = k$ and $u_2 = \frac{2}{3}k$, where k is a constant. Find a and b in terms of k [3]

   (b) Find a formula for u_n in terms of k, giving your answer in its simplest form. [2]

   (c) Determine, with a reason, if the series $\sum_{r=1}^{n} u_r$ converges. [1]

6. **[IJC Prelims 17]**

   A sequence $u_1, u_2, u_3, \ldots$ is such that $u_n = \dfrac{1}{2n^2(n-1)^2}$ and

$$u_{n+1} = u_n - \frac{2}{n(n-1)^2(n+1)^2} \text{ for all } n \geq 2.$$

   (a) Find $\sum_{n=2}^{N} \dfrac{2}{n(n-1)^2(n+1)^2}$. [3]

   (b) Explain why $\sum_{n=2}^{N} \dfrac{2}{n(n-1)^2(n+1)^2}$ is a convergent series, and state the value of the sum to infinity. [2]

   (c) Using your answer to (a), find $\sum_{n=1}^{N} \dfrac{2N}{(n+1)n^2(n+2)^2}$. [2]

7. **[TPJC Prelims 17]**

(a) Express $\dfrac{1}{r^2-1}$ in partial fractions and deduce that

$$\frac{1}{r(r^2-1)}=\frac{1}{2}\left(\frac{1}{r(r-1)}-\frac{1}{r(r+1)}\right).$$

[2]

(b) Hence, find the sum S_n, of the first n terms of the series

$$\frac{1}{2\times 3}+\frac{1}{3\times 8}+\frac{1}{4\times 15}+\cdots$$

[4]

(c) Explain why the series converges, and write down the value of the sum to infinity. [2]

(d) Find the smallest value of n for which S_n is smaller than the sum to infinity by less than 0.0025. [3]

8. **[TJC Prelims 17]**
Given that $\sin[(n+1)x]-\sin[(n-1)x]=2\cos nx\sin x$, show that

$$\sum_{r=1}^{n}\cos rx=\frac{\sin(n+\frac{1}{2})x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}.$$

[4]

Hence express

$$\cos^2\left(\frac{x}{2}\right)+\cos^2(x)+\cos^2\left(\frac{3x}{2}\right)+\ldots+\cos^2\left(\frac{11x}{2}\right)$$

in the form $a\left(\dfrac{\sin bx}{\sin cx}+d\right)$, where a, b, c and d are real numbers. [3]

Answers

1. $\frac{1}{2}\sin(n+\frac{1}{4})\pi - \frac{1}{2\sqrt{2}}$.

2. $\frac{1}{6} - \frac{1}{2(4N+3)}$.

3. (a) $\dfrac{1}{(r+1)!} - \dfrac{1}{(r+2)!}$.

   (b) $\dfrac{1}{3}\left(\dfrac{1}{2} - \dfrac{1}{(n+2)!}\right)$.

   (c) $\frac{1}{3}$.

4. $(n+1)!n - 2$.

5. (a) $a = \frac{2}{3}k, b = 2k$.

   (b) $u_n = \frac{2k}{n!}\left(\frac{n}{n+1}\right)$.

   (c) It converges since $S_n \to 2k$ as $n \to \infty$.

6. (a) $\frac{1}{8} - \frac{1}{2N^2(N+1)^2}$.

   (b) $\frac{1}{8}$.

   (c) $\frac{N}{8}\left(1 - \frac{4}{(N+1)^2(N+2)^2}\right)$.

7. (b) $\frac{1}{4} - \frac{1}{2(n+1)(n+2)}$.

   (c) As $n \to \infty$, $\frac{1}{2(n+1)(n+2)} \to 0$ so $\frac{1}{4} - \frac{1}{2(n+1)(n+2)}$. Sum to infinity $= \frac{1}{4}$.

   (d) 13.

8. $a = \frac{1}{4}, b = \frac{23}{2}, c = \frac{1}{2}, d = 21$.