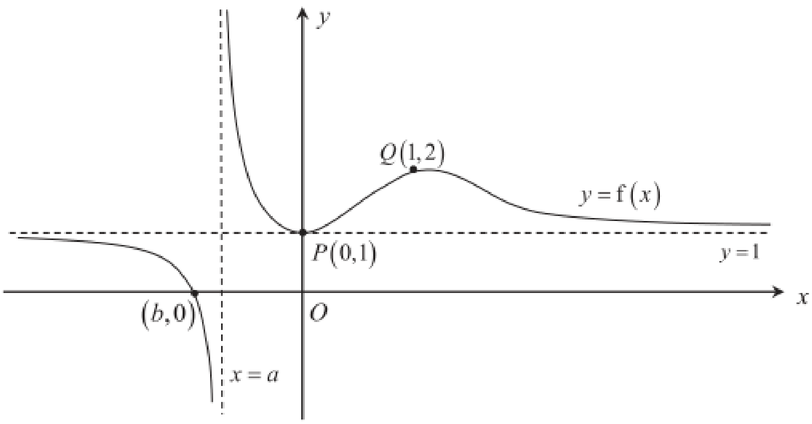
Curves and Transformations Problem Set

1. **[ACJC Prelims 17 (modified)]**
   The diagram shows the graph of $y = f(x)$.

   The graph passes through the point $(b, 0)$ and has turning points at $P(0, 1)$ and $Q(1, 2)$. The lines $y = 1$ and $x = a$, where $b < a < -\frac{1}{2}$, are asymptotes to the curve.

   On separate diagrams, sketch the graphs of

   (a) $y = f\left(\dfrac{x-1}{2}\right)$, [3]

   (b) $y = f'(x)$, [3]

   (c) $y = \dfrac{1}{f(x)}$, [3]

   labelling, in terms of a and b where applicable, the exact coordinates of the points corresponding to P and Q, and the equations of any asymptotes.

2. **[AJC Prelims 17]**

The curve C has equation $y = \dfrac{4x^2 - kx + 2}{x - 2}$, where k is a constant.

(a) Show that c has stationary points when $k < 9$. [3]

(b) Sketch the graph of C for the case where $6 < k < 9$, clearly indicating any asymptotes and points and intersection with the axes. [4]

(c) Describe a sequence of transformations which transforms the graph of $y = 2x + \dfrac{1}{x}$ to the graph of $y = \dfrac{4x^2 - 8x + 2}{x - 2}$. [3]

(d) By drawing a suitable graph on the same diagram as the graph of C, solve the inequality

$$\frac{4x^2 - 8x + 2}{x - 2} > \frac{1}{x^2}.$$

[3]

3. **[CJC Prelims 17]**

A parabola, P with equation $(y - a)^2 = ax$, where a is a constant, undergoes, in succession, the following transformations:

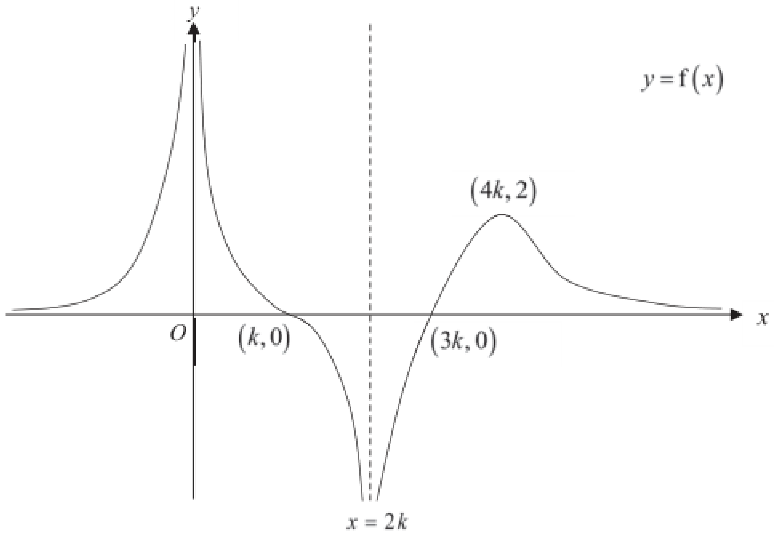
- (A): A translation of 2 units in the positive x-direction,
- (B): A scaling parallel to the y-axis by a factor of $\frac{1}{3}$.

The resulting curve Q passes through the point with coordinates $\left(2, \dfrac{4}{3}\right)$.

(a) Show that $a = 4$. [3]

(b) Find the range of values of k for which the line $y = kx$ does not meet P. [3]

4. **[CJC Prelims 17 (modified)]**
The diagram shows the sketch of the graph of $y = f(x)$ for $k > 0$. The curve passes through the points with coordinates $(k, 0)$ and $(3k, 0)$, and has a maximum point with coordinates $(4k, 2)$. The asymptotes are $x = 0$, $x = 2k$ and $y = 0$.



Sketch, on separate diagrams, the graph of

(a) $y = f(-x - k)$, [2]

(b) $y = f(|x|)$, [2]

(c) $y = \dfrac{1}{f(x)}$, [3]

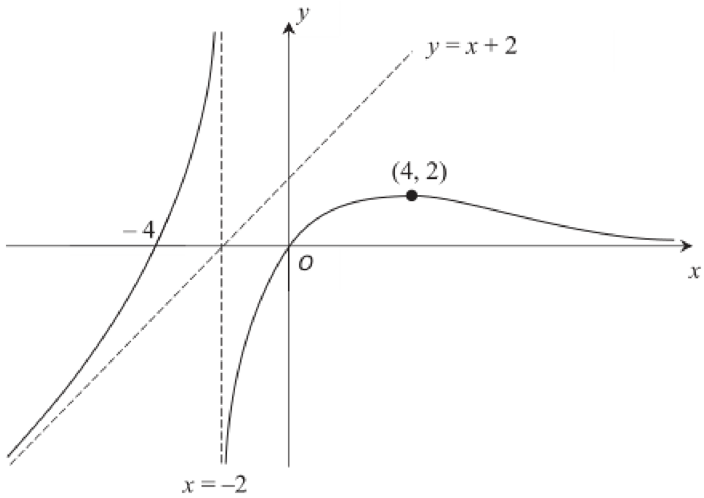
showing clearly, in terms of k, the equations of any asymptote(s), the coordinates of any turning point(s) and any point(s) where the curve crosses the x- and y- axes.

5. **[DHS Prelims 17 (modified)]**

(a) State a sequence of transformations that transform the graph of $x^2 + \frac{1}{3}(y-2)^2 = 1$ to the graph of $(x-2)^2 + y^2 = 1$. [3]

(b) Sketch the graph of $x^2 + \frac{1}{3}(y-2)^2 = 1$, showing clearly its relevant features. [3]

(c) The diagram shows the curve $y = f(x)$. It has a maximum point at $(4, 2)$ and intersects the x-axis at $(-4, 0)$ and the origin. The curve has asymptotes $x = -2$, $y = 0$ and $y = x + 2$.



Sketch, on separate diagrams, the graphs of

i. $y = |f(x)|$, [1]

ii. $y = f'(x)$, [3]

iii. $y = \dfrac{1}{f(x)}$, [3]

including the coordinates of the points where the graphs cross the axes, the turning points and the equations of any asymptotes, where appropriate.

6. **[DHS Prelims 17]**
The curve C with equation

$$y = \frac{x^2 + (a-1)x - a - a}{x-1},$$

where a is a constant, has an oblique asymptote $y = x + 1$.

(a) Show that $a = 1$. Hence sketch C, giving the equations of any asymptotes and the exact coordiantes of any points of intersection with the axes. [3]

(b) ** The region bounded by C for $x > 1$ and the lines $y = x + 1$, $y = 2$ and $y = 4$ is rotated through 2π radians about the line $x = 1$. By considering a translation of C or otherwise, find the volume of the solid of revolution formed. [5]

7. **[HCI Prelims 17 (modified)]**
A curve C_1 has equation $y = \dfrac{ax^2 - bx}{x^2 - c}$, where a, b and c are constants. It is given that C_1 passes through the point $\left(3, \frac{9}{5}\right)$ and two of its asymptotes are $y = 2$ and $x = -2$.

(a) Find the values of a, b and c. [3]

For the rest of the question, use $a = 2, b = 3$ and $c = 4$.

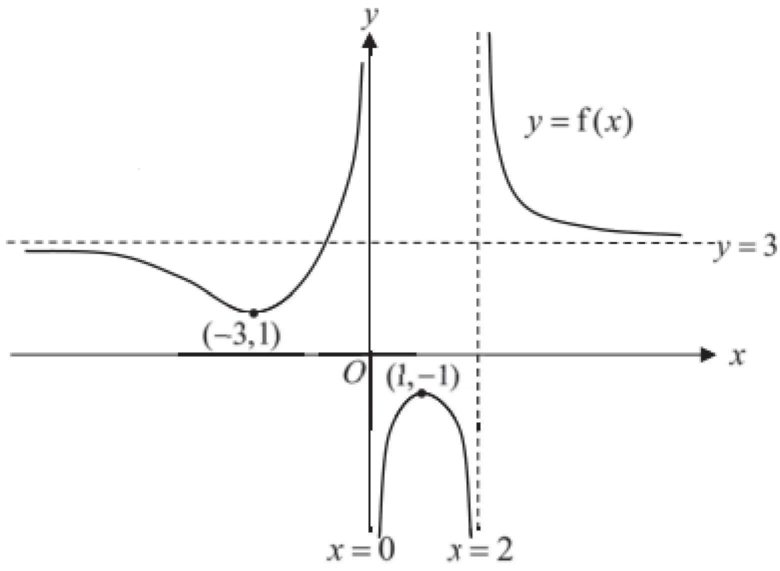
(b) Using an algebraic method, find the exact set of values of y that C_1 cannot take. [3]

(c) Sketch C_1, showing clearly the equations of the asymptotes and the coordinates of the turning points. [3]

(d) It is given that the equation
$e^y = x - r$, where $r \in \mathbb{R}^+$, has exactly oe real root. State the range of values of r. [1]

(e) The curve C_2 has equation $y = 2 + \dfrac{3x+5}{x^2 - 2x - 3}$. State a sequence of transformations which transforms C_1 to C_2. [3]

8. [**TJC Prelims 17 (modified)**]

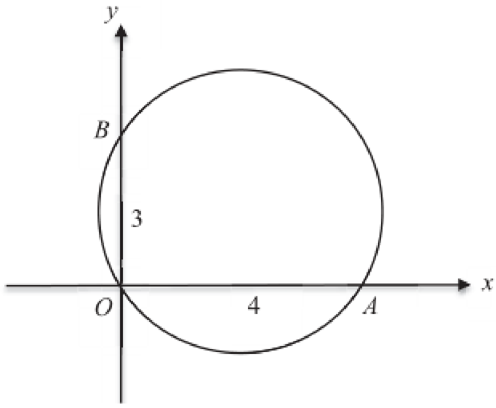


(a) The graph of $y = f(2-x)$ is obtained when the graph of $y = f(x)$ undergoes a sequence of transformations. Describe the sequence of transformations. [2]

(b) Sketch the graph of $y = f'(x)$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

(c) Sketch the graph of $y = \dfrac{1}{f(x)}$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

9. **[TJC Prelims 17]**



The diagram shows a circle C which passes through the origin O and the points A and B. It is given that $OA = 4$ units and $OB = 3$ units.

(a) Show that the coordinates of the centre of C is $\left(2, \dfrac{3}{2}\right)$. Hence write down the equation of C in the form $(x-2)^2 + (y - \frac{3}{2})^2 = r^2$, where r is a constant to be determined. [2]

(b) By adding a suitable line to the diagram above, find the range of values of m for which the equation $mx - \dfrac{3}{2} = \sqrt{\dfrac{25}{4} - (x-2)^2}$ has a solution. [4]

10. **[TPJC Prelims 17]**

The function p is defined by $p : x \mapsto \dfrac{1-x^2}{1+x^2}$, $x \in \mathbb{R}$.

(a) Show algebraically the range of p, showing your working clearly. [3]

(b) Show that $p(x) = p(-x)$ for all $x \in \mathbb{R}$. [1]

It is given that $q(x) = p\left(\frac{1}{2}x - 4\right)$, $x \in \mathbb{R}$.

(c) State a sequence of transformations that will transform the graph of p on to the graph of q. Hence state the line of symmetry for the graph of q. [3]

11. The curve of $y = f(x)$ undergoes the following sequence of transformations:

- (A): A translation 2 units in the positive x-axis direction.
- (B): A scaling parallel to the x-axis with scale factor 3.
- (C): A reflection in the x-axis.
- (D): A translation 1 units in the negative y-axis direction.

The equation of the resulting curve is $y = \dfrac{2-3x}{(3x-2)^2+1}$.

Determine an expression for $f(x)$. [4]

12. Sketch the curves described by the following equations on separate diagrams. Show clearly the asymptotes (together with their equations) and the stationary points (including their coordinates), where applicable.

(a) $(x-2)^2 - \dfrac{(y+3)^2}{4} = 1,$ [3]

(b) $y^2 - 6y - 9x^2 = 0.$ [3]

Answers

2. (c) Translate the graph 2 units in the direction of the positive x-axis.
   Scale the resulting graph parallel to the y-axis, by a scale factor of 2.
   Translate the resulting graph by 8 units in the direction of the positive y-axis.

   (d) $0.805 < x < 1.69$ or $x > 2$.

3. $k < -\frac{1}{4}$.

5. Translate the graph 2 units in the positive x-direction.
   Translate the resulting graph 2 units in the negative y-direction.
   Scale the resulting graph by a factor of $\frac{1}{\sqrt{3}}$ parallel to the y-axis.

7. (a) $a = 2, b = 3$ and $c = 4$.

   (b) $\{y \in \mathbb{R} : 1 - \frac{\sqrt{7}}{4} < y < 1 + \frac{\sqrt{7}}{4}\}$.

   (d) $r \geq 2$.

   (e) Translation of C_1 1 unit in the negative x-direction.
   Reflection of the resulting curve in the y-axis.

8. Translation of 2 units in the negative x-direction followed by a reflection about the y-axis.

9. (a) $r = \frac{5}{2}$.

   (b) $m \leq -3$ or $m \geq \frac{1}{3}$.

10. (a) $-1 < y \leq 1$.

    (c) Translation by 4 units in the positive x-direction following by a stretch of factor 2 parallel to the x-axis.
    $x = 8$.

11. $y = -1 + \frac{x}{x^2+1}$.