Differential Equations Problem Set

1. [**ACJC Prelims 17**]

   (a) Show that for any real constant k,

   $$\int t^2 \mathrm{e}^{-kt}\,\mathrm{d}t = -\mathrm{e}^{-kt}\left(\frac{a}{k}t^2 + \frac{b}{k^2}t + \frac{c}{k^3}\right) + D,$$

   where D is an arbitrary constant, and a, b and c are constants to be determined. [3]

   On the day of the launch of a new mobile game, there were 100,000 players. After t months, the number of players on the game is x, in hundred thousands, where x and t are continuous quantities. It is known that, on average, one player recruits 0.75 players into the game per month, while the number of players who leave the game per month is proportional to t^2.

   (b) Write down a differential equation relating x and t. [1]

   (c) Using the substitution $x = u\mathrm{e}^{\frac{3}{4}t}$, show that the differential equation in (b) can be reduced to

   $$\frac{\mathrm{d}u}{\mathrm{d}t} = -pt^2\mathrm{e}^{-\frac{3}{4}t},$$

   where p is a positive constant.
   Hence solve the differential equation in (b), leaving your answer in terms of p. [5]

   (d) For $p = \frac{1}{3}$, find the maximum number of players on the game, and determine if there will be a time when there are no players on the game. [2]

   (e) Find the range of values of p such that the game will have no more players after some time. [2]

2. [**AJC Prelims 17**]
   Show that the differential equation

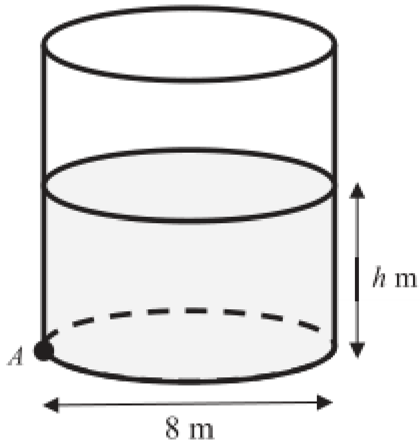
   $$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3xy}{1-3x^2} - x + 1 = 0$$

   may be reduced by means of the substitution $y = u\sqrt{1-3x^2}$ to

   $$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x-1}{\sqrt{1-3x^2}}.$$

   Hence find the general solution for y in terms of x. [5]

3. **[CJC Prelims 17]**
The figure shows a cylindrical water tank with base diameter 8 metres. Water is flowing into the tank at a constant rate of 0.36π m^3/min. At time t minutes, the depth of water in the tank is h metres. However, the tank has a small hole at point A located at the bottom of the tank. Water is leaking from point A at a rate of $0.8\pi h$ m^3/min.



(a) Show that the depth, h metres, of the water in the tank at time t minutes satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{400}(9 - 20h).$$

[3]

(b) Given that $h = 0.4$ when $t = 0$, find the particular solution of the above differential equation in the form $h = f(t)$. [6]

(c) Explain whether the tank will be emptied. [1]

(d) Sketch the part of the curve with the equation found in part (b) which is relevant in this context. [2]

4. **[DHS Prelims 17]**
The variables y and x satisfy the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - \ln x}{x \ln x + 2x^2}.$$

(a) Show that the substitution $u = \frac{\ln x}{x}$ reduces the differential equation to

$$\frac{\mathrm{d}u}{\mathrm{d}y} = u + 2.$$

Given that $y = 0$ when $x = 1$, show that $y = \ln\left(\frac{\ln x}{2x} + 1\right)$. [6]

The curve C has equation $y = \ln\left(\frac{\ln x}{2x} + 1\right)$. It is given that C has a maximum point and two asymptotes $y = a$ and $x = b$.

(b) Find the exact coordinates of the maximum point. [2]

(c) Explain why $a = 0$. You may assume that as $x \to \infty$, $\frac{\ln x}{x} \to 0$. [1]

(d) Determine the value of b, giving your answer correct to 4 decimal places. [2]

(e) Sketch C. [2]

5. **[HCI Prelims 17]**
Food energy taken in by a man goes partly to maintain the healthy functioning of his body and partly to increase body mass. The total food energy intake of the man per day is assumed to be a constant denoted by I (in joules). The food energy required to maintain the healthy functioning of his body is proportional to his body mass M (in kg). The increase of M with respect to time t (in days) is proportional to the energy not used by his body. If the man does not eat for one day, his body mass will be reduced by 1%.

   (a) Show that I, M and t are related by the following differential equation:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{I - aM}{100a}, \quad \text{where } a \text{ is a constant.}$$

   State an assumption for this model to be valid. [3]

   (b) Find the total food energy intake per day, I, of the man in terms of a and M if he wants to maintain a constant body mass. [2]

It is given that the man's initial mass is 100kg.

   (c) Solve the differential equation in part (a), giving M in terms of I, a and t. [3]

   (d) Sketch the graph of M against t for the case where $I > 100a$. Interpret the shape of the graph with regard to the man's food energy intake. [3]

   (e) If the man's total food energy intake per day is $50a$, find the time taken in days for the man to reduce his body mass from 100kg to 90kg. [2]

6. **[IJC Prelims 17]**
A population of a certain organism grows from an initial size of 5. After 5 days, the size of the population is 20, and after t days, the size of the population is M. The rate of growth of the population is modelled as being proportional to $(100^2 - M^2)$.

 (a) Write down a differential equation modelling the population growth and find M in terms of t. [6]

 (b) Find the size of the population after 15 days, giving your answer correct to the nearest whole number. [2]

 (c) Find the least number of days required for the population to exceed 80. [2]

7. **[TPJC Prelims 17]**
A drug is administered by an intravenous drip. The drug concentration, x, in the blood is measured as a fraction of its maximum level. The drug concentration after t hours is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(1 + x - 2x^2),$$

where $0 \leq x < 1$, and k is a positive constant. Initially, $x = 0$.

 (a) Find an expression for x in terms of k and t. [5]

After one hour, the drug concentration reaches 75% of its maximum level.

 (b) Show that the exact value of k is $\frac{1}{3}\ln 10$, and find the time taken for the drug concentration to reach 90% of its maximum level. [3]

A second model is proposed with the following differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sin^2\left(\frac{1}{2}t\right),$$

where x is the drug concentration, measured as a fraction of its maximum level, in the blood after t hours. Initially, $x = 0$.

 (c) Find an equation for x in terms of t. [3]

 (d) Explain, with the aid of a sketch, why this proposed second model is inappropriate. [2]

8. **[TJC Prelims 17]**
A hot air balloon rises vertically upwards from the ground as the balloon operator intermittently fires and turns off the burner. At time t minutes, the balloon ascends at a rate inversely proportional to $t+\lambda$, where λ is a positive constant. At the same time, due to atmospheric factors, the balloon descends at a rate of 2 km per minute. It is also known that initially the rate of change of the height of the balloon is 1 km per minute.

   (a) Find a differential equation expressing the relation between H and t, where H km is the height of the hot air balloon above ground at time t minutes. Hence solve the differential equation and find H in terms of t and λ. [7]

   Using $\lambda = 15$,

   (b) find the maximum height of the balloon above ground in exact form. [3]

   (c) Find the total vertical distance travelled by the balloon when $t = 8$. [3]

   (d) Can we claim that the rate of change of the height of the balloon above the ground is decreasing? Explain your answer. [2]

9. The variables x, y and z are connected by the following differential equations.

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 2 - 5z \qquad (1)$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = z \qquad (2)$$

   (a) Given that $z > \frac{2}{5}$, solve equation (1) to find z in terms of x. [4]

   (b) Hence find y in terms of x. [2]

Answers

1. (a) $-\mathrm{e}^{-kt}\left(\frac{1}{k}t^2+\frac{2}{k^2}t+\frac{2}{k^3}\right)+D.$

   (b) $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{3}{4}x-pt^2.$

   (c) $x=p\left(\frac{4}{3}t^2+\frac{32}{9}t+\frac{128}{27}\right)+D\mathrm{e}^{\frac{3}{4}t}.$

   (d) 365,000.
   Yes, $x=0$ when $t=4.35$ months.

   (e) $p>\frac{27}{128}=0.211.$

2. $y=-\frac{1}{3}(1-3x^2)-\frac{\sqrt{1-3x^2}}{\sqrt{3}}\sin^{-1}\sqrt{3}x+C\sqrt{1-3x^2}.$

3. (b) $h=\frac{1}{20}\left(9-\mathrm{e}^{-\frac{1}{20}t}\right).$

4. (b) $\left(\mathrm{e},\ln(\frac{1}{2\mathrm{e}}+1)\right).$

   (d) $b=0.4263.$

5. (b) $I=aM.$

   (c) $M=\frac{I}{a}-(\frac{I}{a}-100)\mathrm{e}^{-\frac{t}{100}}.$

   (d) In the long run, the man's body mass will approach $\frac{I}{a}$. (Other interpretations possible)

   (e) $t=-100\ln\frac{4}{5}=22.3$ days.

6. (a) $\frac{\mathrm{d}M}{\mathrm{d}t}=k(100^2-M^2).$
   $M=\frac{100(\frac{21}{19}(\frac{19}{14})^{\frac{t}{5}}-1)}{\frac{21}{19}(\frac{19}{14})^{\frac{t}{5}}+1}.$

   (b) $M=47.$

   (c) 35.

7. (a) $x=\frac{\mathrm{e}^{3kt}-1}{\mathrm{e}^{3kt}+2}.$

   (b) 1.45 hours.

   (c) $x=\frac{1}{2}t-\frac{1}{2}\sin t.$

   (d) The graph shows that as time increases, the drug concentration still continue to increase beyond the maximum level of drug concentration.

8. (a) $H = 3\lambda \ln(\frac{t}{\lambda} + 1) - 2t$.
   (b) $15(3 \ln \frac{3}{2} - 1)$.
   (c) 3.26.
   (d) Yes.
9. (a) $z = \frac{2+A\mathrm{e}^{-5x}}{5}$.
   (b) $y = \frac{2x}{5} - \frac{A\mathrm{e}^{-5x}}{25} + B$.