# Function Problem Set

1. **[ACJC Prelims 17 (modified)]**
   The function $f$ is defined by

$$f : x \mapsto \sin\left(x + \frac{1}{4}\right)\pi - \sin\left(x - \frac{3}{4}\right)\pi, \quad x \in \mathbb{R}, a \le x \le 1.$$

   The function $g$ is defined by

$$g : x \mapsto \frac{2x}{1-x}, \quad x \in \mathbb{R}, x \ge \frac{13}{5}.$$

   (a) Express $f(x)$ as a single trigonometric function in the form $b\cos(x - c)\pi$. Hence state the range of $f$ and sketch the curve when $a = -1$, labelling the exact coordinates of the points where the curve crosses the $x$- and $y$- axes. [4]

   (b) State the least value of $a$ such that $f^{-1}$ exists, and define $f^{-1}$ in similar form. [3]

   (c) When $a = -\frac{13}{4}$, show that $fg$ exists. Find the range of $fg$. [3]

2. **[AJC Prelims 17]**
   The function $f$ is defined by

$$f : x \mapsto \frac{e^x - 1}{e - 1} \quad \text{for } x \in \mathbb{R}.$$

   Sketch the graph of $y = f(x)$ and state the range of $f$. [3]

   Another function $h$ is defined by

$$h : x \mapsto \begin{cases} (x-1)^2 + 1 & \text{for } x \le 1 \\ 1 - \frac{|1-x|}{2} & \text{for } 1 < x \le 4 \end{cases}$$

   Sketch the graph of $y = h(x)$ for $x \le 4$ and explain why the composite function $f^{-1}h$ exists. Hence find the exact value of $(f^{-1}h)^{-1}(3)$. [7]

3. **[DHS Prelims 17]**

(a) Express $\sin x + \sqrt{3}\cos x$ as $R\sin(x+\alpha)$, where $R > 0$ and $\alpha$ is an acute angle. [1]

The function $f$ is defined by

$$f : x \mapsto \sin x + \sqrt{3}\cos x, \quad x \in \mathbb{R}, -\frac{\pi}{3} \leq x \leq \frac{\pi}{6}.$$

(b) Sketch the graph of $y = f(x)$. [2]

(c) Find $f^{-1}(x)$, stating the domain of $f^{-1}$. On the same diagram as in part (b), sketch the graph of $y = f^{-1}(x)$, indicating the equation of the line of symmetry. [4]

(d) ** Using integration, find the area of the region bounded by the graph of $f^{-1}$ and the axes. [3]

The function $g$ is defined by

$$g : x \mapsto |\ln(x+2)|, \quad \text{for } x \in \mathbb{R}, x > -2.$$

(e) Show that the composite function $gf^{-1}$ exists, and find the range of $gf^{-1}$. [3]

4. **[HCI Prelims 17]**

The *floor function*, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to $x$. For example, $\lfloor -2.1 \rfloor = -3$ and $\lfloor 3.5 \rfloor = 3$.

The function $f$ is defined by

$$f(x) = \begin{cases} \lfloor x \rfloor & \text{for } x \in \mathbb{R}, -1 \leq x < 2, \\ 0 & \text{for } x \in \mathbb{R}, 2 \leq x < 3, \end{cases}$$

where $\lfloor x \rfloor$ denoted the greatest integer less than equal to $x$.

It is given that $f(x) = f(x+4)$.

(a) Find the values of $f(-1.2)$ and $f(3.6)$. [2]

(b) Sketch the graph of $y = f(x)$ for $-2 \leq x < 4$. [2]

(c) Hence evaluate $\displaystyle\int_{-2}^{4} f(x)\,\mathrm{d}x$. [1]

5. **[IJC Prelims 17]**

The function $f$ is given by $f : x \mapsto 3 + \dfrac{1}{x-2}$ for $x \in \mathbb{R}, x > 2$.

(a) Find $f^{-1}(x)$ and state the domain of $f^{-1}$. [3]

(b) Explain why the composite function $f^2$ exists. [1]

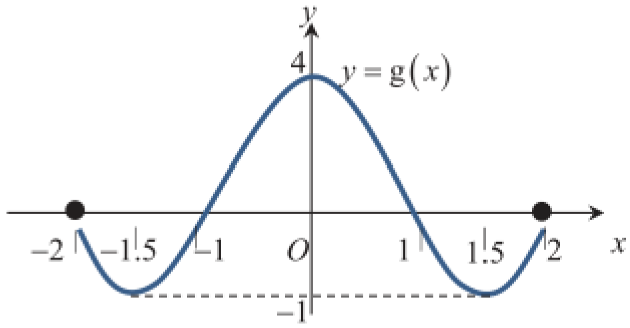
(c) Find the value of $x$ for which $f^2(x) = x$. Explain why this value of $x$ satisfies the equation $f(x) = f^{-1}(x)$. [3]

6. **[TPJC Prelims 17 (modified)]**

The function $f$ is defined by

$$f : x \mapsto (x-k)^2, \quad x < k \text{ where } k > 5.$$

(a) Find $f^{-1}(x)$ and state the domain of $f^{-1}$. [3]




The diagram shows the curve with equation $y = g(x)$ with domain $D_g = [-2, 2]$. The curve crosses the $x$-axis at $x = -2, x = -1, x = 1$ and $x = 2$ and has turning points at $(-1.5, -1)$, $(0, 4)$ and $(1.5, -1)$.

(b) Explain why the composite function $fg$ exists. [2]

(c) Find, in terms of $k$,

i. the value of $fg(-1)$, [1]

ii. the range of $fg$. [2]

7. The functions $f$ and $g$ are defined by

$$f : x \mapsto x^2 + 2x - 3 \qquad \text{for } x \in \mathbb{R}, x < b,$$
$$g : x \mapsto \frac{3x+2}{x-3}, \qquad \text{for } x \in \mathbb{R}, x \neq 3.$$

(a) Determine, with reason, whether $f^{-1}$ exists when

i. $b = -2$,

ii. $b = 2$.

(b) For the value of $b$ in (a) such that $f^{-1}$ exists,

i. solve $f(x) = f^{-1}(x)$ **exactly**. [3]

ii. define $f^{-1}$, stating clearly its domain. [3]

(c) Determine, with reason, whether $gf$ exists when $b = 0$. [1]

(d) Find an expression for $g^{-1}(x)$. [2]

Hence determine

i. $g^2(x)$. [1]

ii. $g^{2017}(8)$. [1]

# Answers

1. (a) $b = 2, c = \frac{1}{4}$.
   $R_f = [-2, 2]$, $(-\frac{1}{4}, 0), (\frac{3}{4}, 0), (0, \sqrt{2})$.
   (b) $a = \frac{1}{4}, f^{-1} : x \mapsto \frac{1}{\pi}\cos^{-1}\frac{x}{2} + \frac{1}{4}, x \in [-\sqrt{2}, 2]$.
   (c) $R_{fg} = [-2, \sqrt{2})$.
2. $1 - \sqrt{\mathrm{e}^2 + \mathrm{e}}$.
3. (a) $2\sin(x + \frac{\pi}{3})$.
   (c) $f^{-1}(x) = -\frac{\pi}{3} + \sin^{-1}(\frac{x}{2})$. $D_{f^{-1}} = R_f = [0, 2]$.
   (d) 1.
   (e) $R_{gf^{-1}} = [0, 0.926]$.
4. $f(-1.2) = f(2.8) = 0$.
   $f(3.6) = f(-0.4) = -1$.
5. (a) $f^{-1}(x) = 2 + \frac{1}{x-3}, x \in \mathbb{R}, x > 3$.
   (c) $x = 3.62$.
6. (a) $f^{-1}(x) = -\sqrt{x} + k$. $D_{f^{-1}} = (0, \infty)$.
   (b) $R_g = [-1, 4] \subseteq (-\infty, k) = D_f$ since $k > 5$. Hence $fg$ exists.
   (c) i. $k^2$.
   ii. $[(4 - k)^2, (1 + k)^2]$.
7. (a) i. Yes. All horizontal lines $y = k, k \in \mathbb{R}$ cuts the curve $y = f(x)$ at most once. Hence $f$ is a one-one function and $f^{-1}$ exists.
   ii. No. The horizontal line $y = 0$ cuts the curve $y = f(x)$ more than once. Hence $f$ is not a one-one function and $f^{-1}$ does not exist.
   (b) i. $\frac{-1-\sqrt{13}}{2}$.
   ii. $f^{-1} : x \mapsto -1 - \sqrt{x + 4}, x \in \mathbb{R}, x > -3$.
   (c) $R_f = (-3, \infty) \nsubseteq D_g = (-\infty, 3) \cup (3, \infty)$.
   (d) $g^{-1}(x) = \frac{3x+2}{x-3}$.
   i. $x$.
   ii. $\frac{26}{5}$.