# Integration Applications Problem Set

1. **[ACJC Prelims 17]**
   Solve the inequality
   $$\frac{1}{x+a} \leq \frac{2a}{x^2 - a^2},$$
   leaving your answer in terms of $a$, where $a$ is a positive real number. [3]

   Hence or otherwise find $\displaystyle\int_{2a}^{4a} \left| \frac{1}{x+a} - \frac{2a}{x^2 - a^2} \right| \mathrm{d}x$ exactly. [4]

2. **[ACJC Prelims 17]**
   A curve has parametric equations
   $$x = \cos t$$
   $$y = \frac{\sin 2t}{2}$$
   where $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$.

   (a) ** Find the equation of the normal to $C$ at the point $P$ with parameter $p$. [2]
   The normal to $C$ at the point where $t = \frac{2\pi}{3}$ cuts the curve again. Find the coordinates of the point of intersection. [2]

   (b) Sketch $C$, clearly labelling the coordinates of the points where the curve crosses the $x$- and $y$- axes. [2]

   (c) Find the cartesian equation of $C$. [2]
   The region bounded by $C$ is rotated through $\pi$ radians about the $x$-axis. Find the exact volume of the solid formed. [3]

3. **[AJC Prelims 17]**

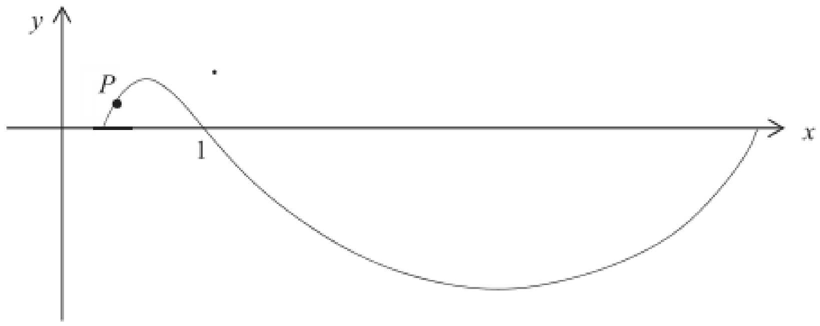
(a) Show by integration that

$$\int \mathrm{e}^{-2x} \sin x \, \mathrm{d}x = -\frac{2}{5}\mathrm{e}^{-2x} \sin x - \frac{1}{5}\mathrm{e}^{-2x} \cos x + A$$

where $A$ is an arbitrary constant [3]

The diagram below shows a sketch of curve $C$, with parametric equations

$$x = \mathrm{e}^{-t}, \qquad y = \mathrm{e}^{-t} \sin t, \qquad -\pi \leq t \leq \pi.$$




Point $P$ lies on $C$ where $t = \frac{\pi}{2}$.

(b) Find the equation of the normal at $P$. [3]

(c) Find the exact area bounded by the curve $C$ for $0 \leq t \leq \pi$, the line $x = 1$ and the normal at $P$. [5]

(d) ** The normal at $P$ cuts the curve $C$ again at two points where $t = q$ and $t = r$. Find the values of $q$ and $r$. [3]

4. **[CJC Prelims 17]**
The region bounded by the curve $y = \frac{1}{\sqrt{x}-2}$, the $x$-axis and the lines $x = 9$ and $x = 16$ is rotated through $2\pi$ radians about the $x$-axis. Use the substitution $t = \sqrt{x}$ to find the exact volume of the solid obtained. [6]

5. [**CJC Prelims 17**]
The function $f$ is defined by

$$f : x \mapsto \frac{1}{3}\tan\left(\frac{x}{3}\right) \text{ for } x \in \mathbb{R}, 0 \leq x < \frac{3\pi}{2}.$$

(a) Sketch the graph of $y = f(x)$, indicating clearly the vertical asymptote. [2]

(b) State the equation of the line of reflection between the graphs of $y = f(x)$ and $y = f^{-1}(x)$, and hence sketch the graph of $y = f^{-1}(x)$ on the same diagram, indicating clearly the horizontal asymptote. [2]

The solutions to the equation $f(x) = f^{-1}(x)$ are $x = 0$ and $x = \alpha$, where $0 < \alpha < \frac{3\pi}{2}$.

(c) Using the diagram drawn, find, in terms of $\alpha$, the area of the region bounded by the curves $y = f(x)$ and $y = f^{-1}(x)$. [5]

Another function $g$ is defined by

$$g : x \mapsto \mathrm{e}^{x} \qquad \text{for } x \in \mathbb{R}, x \geq -2.$$

(d) ** Show that the composite function $gf$ exists and define $gf$ in a similar form. [3]

6. [**DHS Prelims 17 (modified)**]
The curve $C$ with equation

$$y = \frac{x^2 + (a-1)x - a - a}{x - 1},$$

where $a$ is a constant, has an oblique asymptote $y = x + 1$.

(a) Show that $a = 1$. Hence sketch $C$, giving the equations of any asymptotes and the exact coordiantes of any points of intersection with the axes. [3]

(b) The region bounded by $C$ for $x > 1$ and the lines $y = x + 1$, $y = 2$ and $y = 4$ is rotated through $2\pi$ radians about the line $x = 1$. By considering a translation of $C$ or otherwise, find the volume of the solid of revolution formed. [5]

7. **[HCI Prelims 17]**

(a) A curve is defined parametrically by the equations

$$x = \sin t \quad \text{and} \quad y = \cos^3 t, \qquad -\pi \le t \le \pi.$$

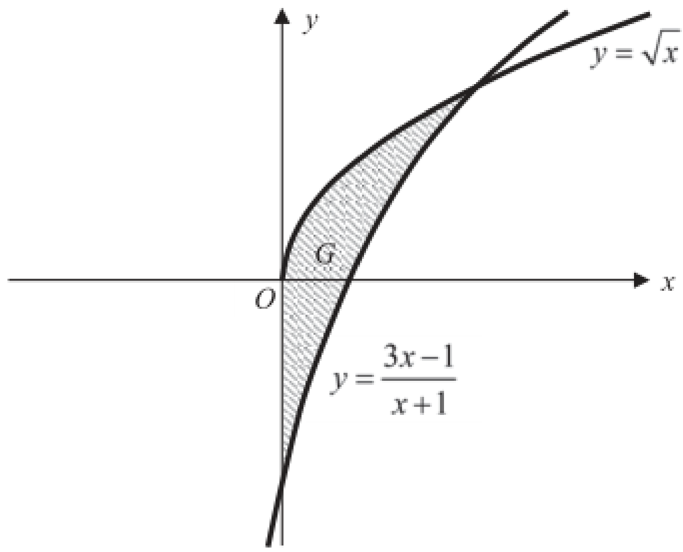
i. Show that the area enclosed by the curve is given by

$$k \int_0^{\frac{\pi}{2}} \cos^4 t \; \mathrm{d}t,$$

where $k$ is a constant to be determined. [3]

ii. Hence find the exact area enclosed by the curve. [3]

(b) In the diagram, the region $G$ is bounded by the curves $y = \frac{3x-1}{x+1}$, $y = \sqrt{x}$ and the $y$-axis.




Find the exact volume of the solid generated when $G$ is rotated about the $y$-axis through $2\pi$ radians. [6]

8. **[IJC Prelims 17]**
Given that $f(x) = \sin 2x + \cos 2x$, express $f(x)$ as $R\sin(2x + \alpha)$, where $R > 0$, $0 < \alpha < \frac{\pi}{2}$ and $R$ and $\alpha$ are constants to be found. [2]

   (a) ** Describe a sequence of transformations that transforms the graph of $y = \sin x$ onto the graph of $y = f(x)$. [3]

   (b) Sketch the graph of $y = f(x)$ for $0 \le x \le \frac{3\pi}{8}$, indicating clearly the exact coordinates of the maximum point and the end points of the graph. [3]

   (c) The region bounded by the curve $y = f(x)$, the line $x = \frac{\pi}{8}$ and both axes is rotated about the $y$-axis through $2\pi$ radians. Find the volume of the solid of revolution correct to 4 decimal places. [4]

9. **[IJC Prelims 17]**
It is given that a curve $C$ has parametric equations

$$x = t^2 - t, \quad y = \frac{1}{t^2+1} \quad \text{for} \quad -2 \le t < 2.$$

   (a) Sketch $C$, indicating clearly the coordinates of the end points and the points where $C$ cuts the $y$-axis. [4]

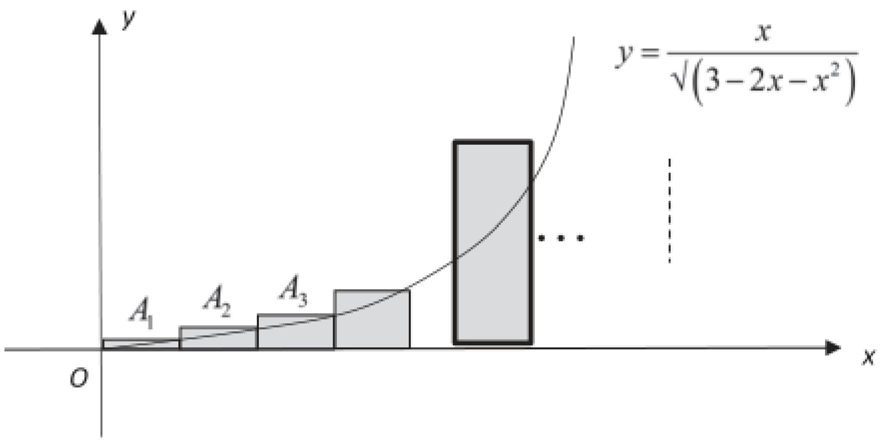
   (b) Find the equation of the tangent to $C$ that is parallel to the $y$-axis. [4]

   (c) Express the area of the region bounded by $C$, the tangent found in part (b) and both axes, in the form

$$\int_a^b f(t)\,\mathrm{d}t,$$

   where the function $f$ and the constants $a$ and $b$ are to be determined. Hence find this area, leaving your answer in exact form. [5]

10. **[TPJC Prelims 17 (modified)]**




The diagram shows the curve with equation $y = \dfrac{x}{\sqrt{3-2x-x^2}}$ for $0 \le x < 1$. The region bounded by the curve, the $x$-axis and the line $x = k$, $0 < k < 1$ is denoted by $S$. It is given that $n$ rectangles of equal width are drawn between $x = 0$ and $x = k$.

(a) Show that the area of the first rectangle, $A_1 = \dfrac{k^2}{n\sqrt{3n^2 - 2nk - k^2}}$. [1]

(b) Show that the total area of all the rectangles i

$$\sum_{r=1}^{n} \frac{rk^2}{n\sqrt{3n^2 - anrk - br^2k^2}},$$

where $a$ and $b$ are constants to be determined. [2]

(c) It is now given that $k = \sqrt{3} - 1$. Use integration to find the actual area of region $S$. Hence state the exact value of

$$\sum_{r=1}^{\infty} \frac{rk^2}{n\sqrt{3n^2 - anrk - br^2k^2}}.$$

[6]

# Answers

1. $x < -a$ or $a < x \leq 3a$.
   $\ln \frac{75}{64}$.

2. (a) $y = \frac{\sin p}{\cos 2p} x + \frac{1}{2}(\sin 2p - \tan 2p)$.
   $(-0.938, 0.325)$.
   (b) $(0,0), (-1,0)$.
   (c) $y^2 = (1-x^2)x^2$.
   $\frac{2\pi}{15}$.

3. (b) $y = -x + 2\mathrm{e}^{-\frac{\pi}{2}}$.
   (c) $\frac{11}{10}\mathrm{e}^{-\pi} - 2\mathrm{e}^{-\frac{\pi}{2}} + \frac{7}{10}$.
   (d) $q = -1.92, r = -1.01$.

4. $\pi(2\ln 2 + 2)$.

5. (b) $y = x$.
   (c) $\alpha^2 + 2\ln\left(\cos\frac{\alpha}{3}\right)$.
   (d) $gf : x \mapsto \mathrm{e}^{\frac{1}{3}\tan\frac{x}{3}}$ for $x \in \mathbb{R}, 0 \leq x < \frac{3\pi}{2}$.

6. 9.75.

7. (a) i. $k = 4$.
   ii. $\frac{3\pi}{4}$.
   (b) $\frac{29\pi}{5} - 8\pi\ln 2$.

8. $f(x) = \sqrt{2}(2x + \frac{\pi}{4})$.
   (a) A translation $\frac{\pi}{4}$ units in the negative $x$-direction.
   A scaling with scale factor $\frac{1}{2}$ parallel to the $x$-axis.
   A scaling with scale factor $\sqrt{2}$ parallel to the $y$-axis.
   (b) 0.6506.

9. (b) $x = -\frac{1}{4}$.
   (c) $\ln\frac{8}{5} - \frac{\pi}{4} + \tan^{-1}\frac{1}{2}$.

10. $\sqrt{3} - 1 - \frac{\pi}{6}$.