Complex Numbers Problem Set

1. **[ACJC Prelims 17]**

   (a) Given that $2z + 1 = |w|$ and $2w - z = 4 + 8i$, solve for w and z. [5]

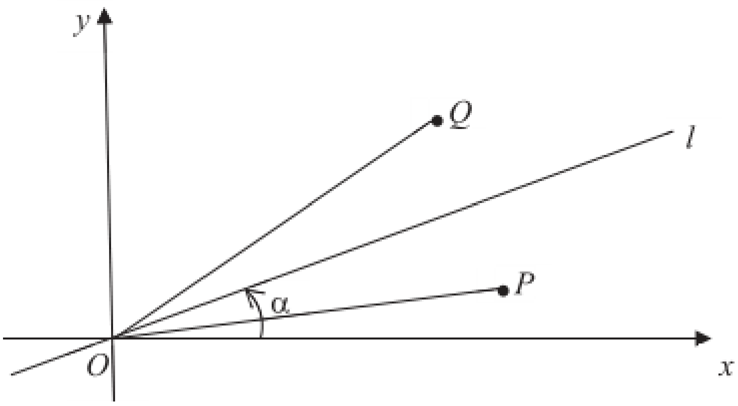
   (b) Find the exact values of x and y, where $x, y \in \mathbb{R}$, such that $2\mathrm{e}^{-\left(\frac{3+x+iy}{i}\right)} = 1 - i$. [4]

2. **[ACJC Prelims 17]**
   Given that $1 + i$ is a root of the equation

   $$z^3 - 4(1+i)z^2 + (-2+9i)z + 5 - i = 0,$$

   find the other roots of the equation. [4]

3. **[AJC Prelims 17]**
   The diagram below shows the line l that passes through the origin and makes an angle α with the positive real axis, where $0 < \alpha < \frac{\pi}{2}$.



   Point P represents the complex number z_1 where $0 < \arg z_1 < \alpha$ and the length of OP is r units. Point P is reflected in line l to produce point Q, which represents the complex number z_2.

   Prove that $\arg z_1 + \arg z_2 = 2\alpha$. [2]

   Deduce that $z_1 z_2 = r^2(\cos 2\alpha + i \sin 2\alpha)$. [1]

   Let R be the point that represents the complex number $z_1 z_2$. Given that $\alpha = \frac{\pi}{4}$, write down the cartesian equation of the locus of R as z_1 varies. [2]

4. [**AJC Prelims 17**]
The polynomial $P(z)$ has real coefficients. The equation $P(z) = 0$ has a root $re^{i\theta}$, where $r > 0$ and $0 < \theta < \pi$. Write down a second root in terms of r and θ, and hence show that a quadratic factor of $P(z)$ is $z^2 - 2rz\cos\theta + r^2$. [2]

Let $P(z) = z^3 + az^2 + 15z + 18$ where a is a real number. One of the roots of the equation $P(z) = 0$ is $3e^{i\left(\frac{2\pi}{3}\right)}$. By expressing $P(z)$ as a product of two factors with real coefficients, find a and the other roots of $P(z) = 0$. [4]

Deduce the roots of the equation $18z^3 + 15z + az + 1 = 0$. [2]

5. [**CJC Prelims 17**]

(a) The complex number z and w satisfy the simultaneous equations

$$z + w^* + 5i = 10 \quad \text{and} \quad |w|^2 = z + 18 + i.$$

Find z and w. [4]

(b) i. It is given that $2 + i$ is a root of the equation

$$z^2 - 5z + 7 + i = 0.$$

Find the second root of the equation in cartesian form, showing your working clearly. [2]

ii. Hence find the roots of the equation $-iw^2 + 5w + 7i - 1 = 0$. [2]

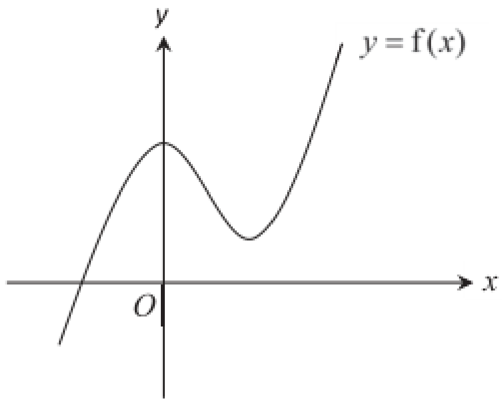
(c) The complex number z is given by $z = -a + ai$, where a is a positive real number.

i. It is given that $w = -\dfrac{\sqrt{2}z^*}{z^4}$. Express w in the form $re^{i\theta}$, in terms of a, where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

ii. Find the two smallest positive whole number values of n such that $\text{Re}(w^n) = 0$. [3]

6. [**DHS Prelims 17 (modified)**]
Do not use a graphic calculator in answering this question.

(a) It is given that $f(x)$ is a cubic polynomial with real coefficients. The diagram shows the curve with equation $y = f(x)$. What can be said about all the roots of the equation $f(x) = 0$? [2]



(b) The complex number z is given by $z = 1 + \mathrm{e}^{i\alpha}$.

i. Show that z can be expressed as $2\cos\left(\frac{\alpha}{2}\right)\mathrm{e}^{i\left(\frac{\alpha}{2}\right)}$. [2]

ii. Given that $\alpha = \frac{\pi}{3}$ and $w = -1 - \sqrt{3}i$, find the exact modulus and argument of $\left(\frac{z}{w^3}\right)^*$. [5]

7. [**HCI Prelims 17**]
The complex number z is given by $z = r\mathrm{e}^{i\theta}$, where $r > 0$ and $0 \leq \theta \leq \pi$. It is given that the complex number $w = (-\sqrt{3} - i)z$.

(a) Find $|w|$ in terms of r, and $\arg w$ in terms of θ. [2]

(b) Given that $\frac{z^5}{w^*}$ is purely imaginary, find the three smallest values of θ in terms of π. [5]

8. [**HCI Prelims 17**]
The complex numbers z and w satisfy the following equations

$$2z + 3w = 20,$$
$$w - zw^* = 6 + 22i.$$

(a) Find z and w in the form $a + bi$, where a and b are real, $a \neq 0$. [5]

(b) Show z and w on a single Argand diagram, indicating clearly their modulus. State the relationship between z and w with reference to the origin O. [2]

9. **[IJC Prelims 17 (modified)]**
A graphic calculator is **not** to be used in answering this question.

The equation $w^3+pw^2+qw+30=0$, where p and q are real constants, has a root $w=2-i$. Find the values of p and q, showing your working. [3]

10. **[IJC Prelims 17]**
The complex number z is such that $|z|=1$ and $\arg z=\theta$, where $0<\theta<\frac{\pi}{4}$.

    (a) Mark a possible point A representing z on an Argand diagram. Hence mark the points B and C representing z^2 and $z+z^2$ respectively on the same Argand diagram corresponding to point A. [3]

    (b) State the geometrical shape of $OACB$. [1]

    (c) Express $z+z^2$ in polar form, $p\cos(q\theta)\left[\cos(k\theta)+i\sin(k\theta)\right]$ where p, q and k are constants to be determined. [2]

11. **[TPJC Prelims 17]**
It is given that $z=-1-i\sqrt{3}$.

    (a) Given that $\dfrac{(iz)^n}{z^2}$ is purely imaginary, find the smallest positive integer n [4]

    The complex umber w is such that $|wz|=4$ and $\arg\left(\dfrac{w^*}{z^2}\right)=-\dfrac{5\pi}{6}$.

    (b) Find the value of $|w|$ and the exact value of $\arg(w)$ in terms of π. [3]

    On an Argand diagram, points A and B represent the complex numbers w and z respectively.

    (c) Referred to the origin O, find the exact value of the angle OAB in terms of π. Hence or otherwise find the exact value of $\arg(z-w)$ in terms of π. [2]

12. **[TPJC Prelims 17]**
The cubic equation $az^3 - 31z^2 + 212z + b = 0$, where a and b are real numbers, has a complex root $z = 1 - 3i$.

   (a) Explain why the equation must have a real root. [2]

   (b) Find the values of a and b and the real root, showing your working clearly. [5]

13. **[TJC Prelims 17]**

   (a) In an Argand diagram, points P and Q represent the complex numbers $z_1 = 2 + 3i$ and $z_2 = iz_1$.

      i. Find the area of the triangle OPQ, where O is the origin. [2]

      ii. z_1 and z_2 are roots of the equation $(z^2+az+b)(z^2+cz+d) = 0$, where $a, b, c, d \in \mathbb{R}$. Find a, b, c and d. [4]

   (b) Without using a graphing calculator, find in exact form, the modulus and argument of $v^* = \left(\dfrac{\sqrt{3}+i}{-1+i}\right)^{14}$. Hence express v in exponential form. [5]

Answers

1. (a) $z = 2, w = 3 + 4i$.
   (b) $x = -\frac{\pi}{4} - 3, y = \frac{1}{2}\ln 2$.

2. $z = 3 + 2i$ or $z = i$.

3. $x = 0, y > 0$.

4. $a = 5$.
   $3e^{i\left(\frac{2\pi}{3}\right)}, 3e^{i\left(\frac{-2\pi}{3}\right)}, -2 = 2e^{i\pi}$.
   $\frac{1}{3}e^{i\left(\frac{2\pi}{3}\right)}, \frac{1}{3}e^{i\left(-\frac{2\pi}{3}\right)}, -\frac{1}{2}$.

5. (a) $w = 3 + 4i, z = 7 - i$.
   $w = -4 + 4i, z = 14 - i$.
   (b) i. $3 - i$.
   ii. $w = 1 - 2i, w = -1 - 3i$.
   (c) i. $\frac{1}{2a^3}e^{i\left(-\frac{3\pi}{4}\right)}$.
   ii. $2, 6$.

6. (a) There is exactly one real root. The other two roots are complex and they exist as a conjugate pair.
   (b) $\frac{\sqrt{3}}{8}, -\frac{\pi}{6}$.

7. (a) $2r, \theta - \frac{5\pi}{6}$.
   (b) $\frac{\pi}{27}, \frac{4\pi}{27}, \frac{7\pi}{27}$.

8. (a) $w = 6 + 2i, z = 1 - 3i$.
   (b) $\angle WOZ$ is 90°.

9. $p = 2, q = -19$.

10. (b) Rhombus.
    (c) $2\cos\frac{\theta}{2}(\cos\frac{3\theta}{2} + i\sin\frac{3\theta}{2})$.

11. (a) 5.
    (b) $|w| = 2, \arg(w) = \frac{13\pi}{6}$.
    (c) $-\frac{3\pi}{4}$.

12. (a) Since the coefficients are real, complex roots occur in conjugate pairs.

    Since a cubic equation has three roots, the third root must be a real root.

    (b) $a = 25, b = 190, -\frac{19}{25}$.

13. (a) i. $\frac{13}{2}$.

        ii. $a = -4, b = 13, c = 6, d = 13$.

    (b) $v = 2^7 \mathrm{e}^{i\frac{\pi}{6}}$.