# Differentiation Problem Set

1. **[ACJC Prelims 17 (modified)]**
The curve $C$ and the line $L$ have equations $y = x^2$ and $y = \frac{1}{2}x - 2$ respectively.

   (a) The point $A$ on $C$ and the point $B$ on $L$ are such that they have the same $x$-coordinate. Find the coordinates of $A$ and $B$ that gives the shortest distance $AB$. [3]

   (b) A variable point on the curve $C$ with coordinates $(s, s^2)$ starts from the origin $O$ and moves along the curve with $s$ increasing at a rate of 2 units/s. Find the rate of change of the area bounded by the curve, the $y$-axis and the line $y = s^2$, at the instant when $s = \sqrt{2}$. [4]

2. **[AJC Prelims 17 (modified)]**

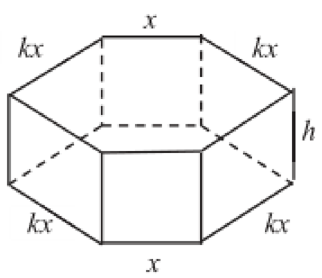



Fig 1

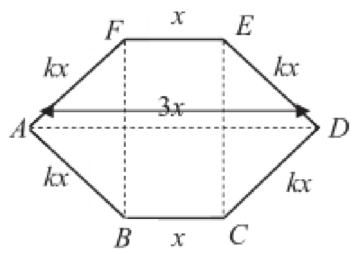



Fig 2

Figure 1 shows a solid metal hexagonal prism of height $h$ cm. Figure 2 shows the hexagonal cross-section $ABCDEF$ of the prism where $AD = 3x$ cm, $BC = FE = x$ cm and the remaining 4 sides are of length $kx$ cm each, where $k$ is a constant.

Show that
$$S = 8x^2\sqrt{k^2 - 1} + 2xh(1 + 2k),$$
where $S$ is the surface area of the hexagonal prism. [3]

If the volume of the prism is fixed at 400 cm$^3$, use differentiation to find, in terms of $k$, the exact value of $x$ that gives a stationary value of $S$. [3]

3. **[AJC Prelims 17]**

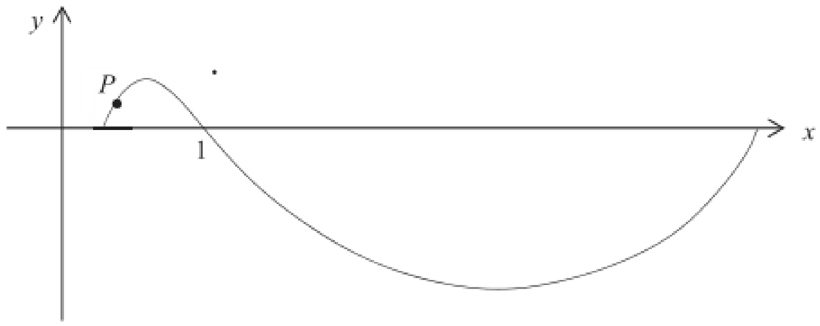
(a) ** Show by integration that

$$\int e^{-2x} \sin x \, dx = -\frac{2}{5}e^{-2x} \sin x - \frac{1}{5}e^{-2x} \cos x + A$$

where $A$ is an arbitrary constant [3]

The diagram below shows a sketch of curve $C$, with parametric equations

$$x = e^{-t}, \qquad y = e^{-t} \sin t, \qquad -\pi \leq t \leq \pi.$$




Point $P$ lies on $C$ where $t = \frac{\pi}{2}$.

(b) Find the equation of the normal at $P$. [3]

(c) ** Find the exact area bounded by the curve $C$ for $0 \leq t \leq \pi$, the line $x = 1$ and the normal at $P$. [5]

(d) The normal at $P$ cuts the curve $C$ again at two points where $t = q$ and $t = r$. Find the values of $q$ and $r$. [3]

4. **[CJC Prelims 17]**
A straight line passes through the point with coordinates $(4, 3)$, cuts the positive $x$-axis at point $P$ and the positive $y$-axis at point $Q$. It is given that $\angle PQO = \theta$, where $0 < \theta < \frac{\pi}{2}$ and $O$ is the origin.

(a) Show that the equation of line $PQ$ is given by $y = (4-x)\cot \theta + 3$
. [2]

(b) By finding an expression for $OP + OQ$, show that as $\theta$ varies, the stationary value of $OP + OQ$ is $a + b\sqrt{3}$, where $a$ and $b$ are constants to be determined. [5]

5. **[CJC Prelims 17]**
A curve $C$ has parametric equations

$$x = \frac{4}{t+1} \quad \text{and} \quad y = t^2 - 3, \quad t \neq -1.$$

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $t$. [2]

(b) Find the equation of the normal to $C$ at $P$ where $x = -2$. [3]

(c) Find the other values of $t$ where the normal at $P$ meets the curve $C$ again. [3]

6. **[DHS Prelims 17]**
Given that $\operatorname{cosec} y = x$ for $0 < y < \frac{\pi}{2}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $y$. Deduce that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{cosec}^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}$$

for $x > 1$. [3]

7. **[HCI Prelims 17]**
A particle moving along a path at time $t$, where $0 < t < \frac{\pi}{3}$, is defined parametrically by

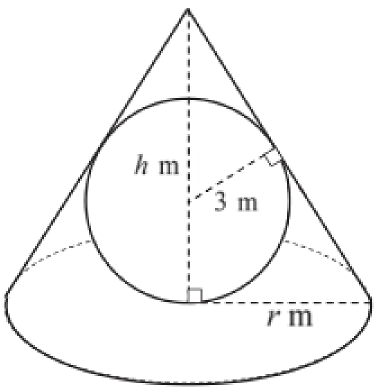
$$x = \cot 3t \quad \text{and} \quad y = 2\operatorname{cosec} 3t + 1.$$

(a) The tangent to the path at the point $P(\cot 3p, 2\operatorname{cosec} 3p + 1)$ meets the $y$-axis at the point $Q$. Show that the coordinates of $Q$ is $(0, 2\sin 3p + 1)$. [4]

(b) The distance of the particle from the point $R(0, 1)$ is denoted by $s$, where $s^2 = x^2 + (y-1)^2$. Find the exact rate of change of the particle's distance from $R$ at time $t = \frac{\pi}{4}$. [4]

8. **[IJC Prelims 17]**
In a distant Northern kingdom of Drivenbell, Elsanna builds a spherical snowball with radius 3 m. The snowball is inscribed in a right conical container of base radius $r$ m and height $h$ m. The container is specially designed to allow the snowball to remain intact with fixed radius 3 m (see diagram).

You may use the fact that the volume of a circular cone with base radius $r$ and height $h$ is $\frac{1}{3}\pi r^2 h$ and the volume and surface area of a sphere of radius $r$ are $\frac{4}{3}\pi r^3$ and $4\pi r^2$ respectively.




(a) By considering the slant height of the cone, show that

$$r = \frac{3h}{\sqrt{h^2 - 6h}}.$$

[3]

(b) Use differentiation to find the values of $h$ and $r$ that give a minimum volume for the container. Find the value of the minimum volume. [6]

The snowball is being removed from the container and it starts to melt under room temperature.

(c) Assuming that the snowball remains spherical as it melts, find the rate of decrease of its volume at the instant when the radius of the sphere is 2.5m, given that the surface area is decreasing at 0.75 $m^2$ per minute at this instant. [5]

9. **[IJC Prelims 17]**
It is given that a curve $C$ has parametric equations

$$x = t^2 - t, \quad y = \frac{1}{t^2+1} \quad \text{for} \quad -2 \leq t < 2.$$

(a) Sketch $C$, indicating clearly the coordinates of the end points and the points where $C$ cuts the $y$-axis. [4]

(b) Find the equation of the tangent to $C$ that is parallel to the $y$-axis. [4]

(c) ** Express the area of the region bounded by $C$, the tangent found in part (b) and both axes, in the form

$$\int_a^b f(t)\,\mathrm{d}t,$$

where the function $f$ and the constants $a$ and $b$ are to be determined. Hence find this area, leaving your answer in exact form. [5]

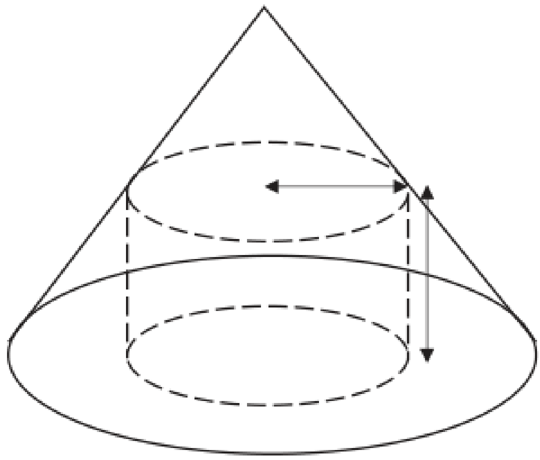
10. **[TJC Prelims 17]**
The curve $C$ has equation $y = \sin 2x + 2\cos x, 0 \leq x \leq 2\pi$.

(a) Using an algebraic method, find the exact $x$-coordinates of the stationary points. You do not need to determine the nature of the stationary points. [3]

(b) Sketch the curve $C$, indicating clearly the coordinates of the turning points and the intersection with the axes. [1]

(c) ** Find the area bounded by the curve $C$ and the line $y = \frac{1}{\pi}x$. [3]

11. **[TPJC Prelims 17]**



A metal cylinder of radius $r$ cm and height $h$ cm is inscribed in a circular cone paperweight of base radius 4 cm and height 6 cm (see diagram). It is determined that the volume of the cylinder, $V$ cm$^3$, should be as large as possible to provide weight to the paperweight. Show that

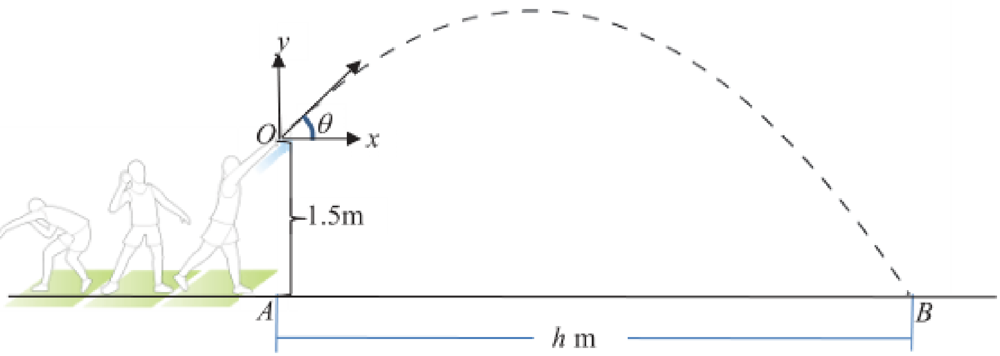
$$V = \frac{4\pi}{9}(36h - 12h^2 + h^3).$$

[2]

Hence find the exact maximum value of $V$. [5]

The metal cylinder is known to expand under heat. An experiment shows that the height of the cylinder is increasing at a rate of 0.04cm s$^{-1}$ at an instant when $h = 1.5$.
Find the rate of change of $V$ at this instant. [2]

12. [**TJC Prelims 17**]




The diagram shows a shot put being projected with a velocity $v$ ms$^{-1}$ from the point $O$ at an angle $\theta$ made with the horizontal. The point $O$ is 1.5m above the point $A$ on the ground. The $x$-$y$ plane is taken to be the plane that contains the trajectory of this projectile motion with $x$-axis parallel to the horizontal and $O$ being the origin. The equation of the trajectory of this projectile motion is known to be

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta},$$

where $g$ ms$^{-2}$ is the acceleration due to gravity.
The constant $g$ is taken to be 10 and the distance between $A$ and $B$ is denoted by $h$ m. Given that $v = 10$, show that $h$ satisfies the equation

$$h^2 - 10h\sin 2\theta - 15\cos 2\theta - 15 = 0.$$

[3]

As $\theta$ varies, $h$ varies. Show that the stationary value of $h$ occurs where $\theta$ satisfies the equation

$$3\tan^2 2\theta - 20\sin 2\theta\tan 2\theta - 20\cos 2\theta - 20 = 0.$$

[5]

Hence find the stationary value of $h$. [2]

# Answers

1. (a) $A(\frac{1}{4}, \frac{1}{16})$, $B(\frac{1}{4}, -\frac{15}{8})$.
   (b) 8 units$^2$/s.

2. $x = \sqrt{\frac{25(1+2k)}{2(k^2-1)}}$.

3. (b) $y = -x + 2e^{-\frac{\pi}{2}}$.
   (c) $\frac{11}{10}e^{-\pi} - 2e^{-\frac{\pi}{2}} + \frac{7}{10}$.
   (d) $q = -1.92, r = -1.01$.

4. $7 + 4\sqrt{3}$.

5. (a) $-\frac{t(t+1)^2}{2}$.
   (b) $y = -\frac{1}{6}x + \frac{17}{3}$.
   (c) $t = -3$ or $t = -0.915$ or $t = 2.91$.

6. $\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \cot y}$.

7. 10 units/s.

8. (b) $h = 12, r = \frac{6}{\sqrt{2}}$. $V = 72\pi$ m$^3$.
   (c) 0.9375 m$^3$ per minute.

9. (b) $x = -\frac{1}{4}$.
   (c) $\ln\frac{8}{5} - \frac{\pi}{4} + \tan^{-1}\frac{1}{2}$.

10. (a) $x = \frac{\pi}{6}, \frac{5\pi}{6}$ or $\frac{3\pi}{2}$.
    (c) 8.92.

11. $V = \frac{128\pi}{9}$.
    $\frac{dV}{dt} = 0.12\pi$ cm$^3$s$^{-1}$.

12. $h = 11.4$.