## 1. Equations and inequalities

## Rational inequalities

## Approach

- Step 1: Make one side of the inequality 0 by addition/subtraction. Do not cross multiply!
- Step 2: Combine into a single fraction and factorize as much as possible.
- Step 3: If a quadratic has no real roots, complete the square to conclude it is always positive/negative.
- Step 4: Use a number line to obtain the solution.



## Modulus inequalities

## Formula

Let $k$ be a positive real number.

| $\|x\|<k$ | $\Rightarrow$ | $-k<x<k$ |
| ---: | :--- | :--- |
| $\|x\|>k$ | $\Rightarrow$ | $x<-k$ or |

## Graphing calculator techniques

## Approach

- We can use the "PlySmlt2" app to solve systems of linear equations.
- The "Plysmlt2" app can also be used to solve polynomials like $2 x^{3}-x^{2}+2 x-1=0$
- We can use the graph of a curve and the "zero" solver to solve equations like $\ln x-x^{2}=0$.
- We can use the graph of two curves and the "intersect" solver to solve equations like $\ln x=x^{2}$.
- For non-rational inequalities like $\mathrm{e}^{x}>3 x$, we can use a GC to sketch two curves and locate the region(s) where one curve is higher than the other.
Alternatively, we can sketch one curve by rearranging the inequality to $\mathrm{e}^{x}-3 x>0$.


## The discriminant

## Approach

- Step 1: Cross multiply and form a quadratic equation in terms of $x$.
- Step 2: For the set of values that $y$ can take, the discriminant $b^{2}-4 a c \geq 0$.


## Example

$$
y=\frac{x+1}{x^{2}+x+1} .
$$

Use an algebraic method to find the set of values that $y$ can take.

$$
\begin{array}{ll}
\text { Step 1: } \\
y=\frac{x+1}{x^{2}+x+1} & \begin{array}{l}
\text { Step 2: } \\
\text { For set of values that } y \text { can take, }
\end{array} \\
\left(x^{2}+x+1\right) y=x+1 & (y-1)^{2}-4(y)(y-1) \geq 0 \\
y x^{2}+y x+y-x-1=0 & -3 y^{2}+2 y+1 \geq 0 \\
y x^{2}+(y-1) x+(y-1)=0 & (3 y+1)(-y+1) \geq 0 \\
\text { Solution: }-\frac{1}{3} \leq y \leq 1
\end{array}
$$

## 2. Functions

## Basics

## Definition

## Set and interval notation

## Example

- We can refer to the set of all real numbers using $\mathbb{R},\{x: x \in \mathbb{R}\}$ or $(-\infty, \infty)$.
- We can refer to all the real numbers from -5 (inclusive) to 3 (non-inclusive) by $\{x \in \mathbb{R}:-5 \leq x<3\}$ or $[-5,3)$.
- We can refer to all the real numbers except 1 by $\mathbb{R} \backslash\{1\}$ or $\{x \in \mathbb{R}: x \neq 1\}$ or $(-\infty, 1) \cup(1, \infty)$.


## Inverse functions

## Theory

- A function has an inverse, denoted by $f^{-1}$, if $f$ is one-one.
- We can determine if $f$ is one-one by employing the horizontal line test.
- To find the rule for $f^{-1}$, we let $y=f(x)$ and make $x$ the subject.
- $D_{f^{-1}}=R_{f}$, $R_{f^{-1}}=D_{f}$.
- The graph of $y=f^{-1}(x)$ can be obtained by reflecting the graph of $y=f(x)$ in the line $y=x$.



## Composite functions

## Theory

- The composite function $f g$ consists of first applying $g$ followed by $f$.
- $f g$ exists if $R_{g} \subseteq D_{f}$. $f g$ does not exist if $R_{g} \nsubseteq D_{f}$.
- $D_{f g}=D_{g}$.
- To find $R_{f g}$, we draw the graphs of $y=f(x)$ and $y=g(x)$ separately. Find $R_{g}$ first, and then use $R_{g}$ as the domain of $f$ to obtain $R_{g f}$.

$R_{g}=[-1, \infty), D_{f}=(0,2)$.
$R_{f}=(-\infty, \ln 2), D_{g}=(-\infty, \infty)$.
$R_{g} \not \subset D_{f} \Rightarrow f g$ does not exist $R_{f} \subseteq D_{g} \Rightarrow g f$ exists $g f(x)=g(\ln x)=(\ln x)^{2}-1$
$D_{g f}=D_{f}=(0, \infty)$
$R_{g f}=[-1, \infty)$


## Special examples

$f f^{-1}(x)=x, \quad f^{-1} f(x)=x, \quad f^{2}(x)=f f(x), \quad$ periodic function: $f(x+a)=f(x)$.

$$
\begin{aligned}
& \text { A piecewise function } \\
& f(x)= \begin{cases}x^{2} & \text { for } x-1 \leq x<1 \\
2.5-x & \text { for } 1 \leq x \leq 2.5 \\
0 & \text { otherwise }\end{cases}
\end{aligned}
$$



## 3. Graphs and transformations



## Conics

$$
\begin{aligned}
& \text { Ellipse } \\
& \frac{(x-h)^{2}}{a^{2}}+\frac{(y-k)^{2}}{b^{2}}=1
\end{aligned}
$$

Center ( $h, k$ ),
horizontal radius $a$,

vertical radius $b$.

$$
\begin{aligned}
& \text { Hyperbola } \\
& \frac{(x-h)^{2}}{a^{2}}-\frac{(y-k)^{2}}{b^{2}}=1 \\
& \frac{(y-k)^{2}}{b^{2}}-\frac{(x-h)^{2}}{a^{2}}=1
\end{aligned}
$$

Center $(h, k)$.


Completing the square is a useful technique to obtain the forms above.

## Basic transformations (translation, scaling, reflection)

| Transformation | Equation |
| :---: | :---: |
| translate $a$ units in positive $x$-axis direction scale with scale factor $b$ parallel to the $x$-axis reflect in $y$-axis | $\begin{gathered} \text { replace } x \text { with } x-a \\ f(x) \rightarrow f(x-a) \\ \text { replace } x \text { with } \frac{x}{b} \\ f(x) \rightarrow f\left(\frac{x}{b}\right) \\ \text { replace } x \text { with }-x \\ f(x) \rightarrow f(-x) \end{gathered}$ |
| translate $A$ units in positive $y$-axis direction scale with scale factor $B$ parallel to $y$-axis reflect in $x$-axis | $\begin{gathered} \text { replace } y \text { with } y-A \\ f(x) \rightarrow f(x)+A \\ \text { replace } y \text { with } \frac{y}{B} \\ f(x) \rightarrow B f(x) \\ \text { replace } y \text { with }-y \\ f(x) \rightarrow-f(x) \end{gathered}$ |

Order matters. For example,
$f(x) \rightarrow f(x+1) \rightarrow f(2 x+1)$ vs $f(x) \rightarrow f(2 x) \rightarrow f(2(x+1))$.

Further transformations $\left(y=\frac{1}{f(x)}, y=f^{\prime}(x)\right)$

## Approach

| $y=f(x)$ | $y=\frac{1}{f(x)}$ | $y=f^{\prime}(x)$ |
| :---: | :---: | :---: |
| horizontal asymptote $y=k$ | horizontal asymptote $y=\frac{1}{k}$ | horizontal asymptote $y=0$ |
| oblique asymptote $y=m x+c$ | horizontal asymptote $y=0$ | horizontal asymptote $y=m$ |
| vertical asymptote $x=a$ | $x$-intercept $(a, 0)$ | vertical asymptote $x=a$ |
| $x$-intercept $(b, 0)$ | vertical asymptote $x=b$ | - |
| $y$-intercept $(0, d)$ | $y$-intercept $\left(0, \frac{1}{d}\right)$ | - |
| max/min point $(A, B)$ | min/max point $\left(A, \frac{1}{B}\right)$ | $x$-intercept $(A, 0)$ |
| $y$ increasing/decreasing | $y$ decreasing/increasing | $y$ positive/negative |
| $y$ positive/negative | $y$ positive/negative | - |
| slope increasing in magnitude | - | $y$ increasing in magnitude |

## 4. Arithmetic and geometric progressions, the sigma notation

## Basics

Theory

- Let $u_{n}$ denote the $n$th term of a sequence.
- Let $S_{n}$ denote the sum of the first $n$ terms of a series $S_{n}=u_{1}+u_{2}+\ldots+u_{n-1}+u_{n}$.
- A sequence converges if $u_{n}$ gets arbitrarily close to a finite number when $n$ gets very large.
We write $u_{n} \rightarrow a$ as $n \rightarrow \infty$, or $\lim _{n \rightarrow \infty} u_{n}=a$. $a$ is called the limit of the sequence.
- A series converges if $S_{n}$ gets arbitrarily close to a finite number when $n$ gets very large.
We write $S_{n} \rightarrow b$ as $n \rightarrow \infty$, or $\lim _{n \rightarrow \infty} S_{n}=S_{\infty}=b$. $b$ is called the limit of the series.
- If a sequence/series does not converge, it is said to diverge
- To recover $u_{n}$ from $S_{n}: u_{n}=S_{n}-S_{n-1}$.

Arithmetic progressions (APs)

## Formula

- Let $a$ be the first term of an AP and $d$ the common difference.
- To prove that a sequence/series is arithmetic, we prove that $u_{n}-u_{n-1}=$ constant
- $u_{n}=a+(n-1) d$
- $S_{n}=\frac{n}{2}(2 a+(n-1) d)=\frac{n}{2}\left(a+u_{n}\right)$


## Method of differences

The method of differences can be used for sums like $\sum_{r=1}^{n} \frac{1}{r(r+1)}$. Partial fractions is often useful.

## Geometric progressions (GPs)

## Formula

- Let $a$ be the first term of an GP and $r$ the common ratio.
- To prove that a sequence/series is geometric, we prove that $\frac{u_{n}}{u_{n-1}}=$ constant.
- $u_{n}=a r^{n-1}$
$S_{n}=\frac{a\left(1-r^{n}\right)}{1-r}=\frac{a\left(r^{n}-1\right)}{r-1}$
- If $-1<r<1$, then a GP converges and sum to infinity $S_{\infty}=\frac{a}{1-r}$.


## Sigma notation

## Theory

- Example: $\sum_{r=3}^{5} f(r)=f(3)+f(4)+f(5)$.
- There are $b-a+1$ terms in $\sum_{r=a}^{b} f(r)$.
- Sum of a constant, $\sum_{r=a}^{b} k=(b-a+1) k$.
- $\sum_{r=a}^{b}(c r+d)$ is an AP, $\sum_{r=a}^{b} c^{r}$ is a GP.
- $\sum_{r=a}^{b}(k f(r) \pm g(r))=k \sum_{r=a}^{b} f(r) \pm \sum_{r=a}^{b} g(r)$.
- $\sum_{r=a}^{c} f(r)=\sum_{r=a}^{b} f(r)+\sum_{r=b+1}^{c} f(r)$ where $a \leq b<c$.


## Change of variable <br> Example <br> $$
\text { Suppose } \sum_{r=1}^{n} r^{2}=\frac{n(n+1)(2 n+1)}{6} \text {. Find } \sum_{r=2}^{n}(r+1)^{2} \text {. }
$$

Step 1a: Replace $r$ with $r-1$.
Step 1b: Lower limit: $r-1=2 \Rightarrow r=3$.
Step 1c: Since lower limit increases by 1, upper limit also increases by 1 .
Step 2: Split up the summation so that the lower limit matches what we know.
Step 3: Apply known formula.

$$
\begin{aligned}
\sum_{r=2}^{n}(r+1)^{2} & =\sum_{r=3}^{n+1} r^{2} \\
& =\sum_{r=1}^{n+1} r^{2}-\sum_{r=1}^{2} r^{2} \\
& =\frac{(n+1)(n+2)(2 n+3)}{6}-\frac{(2)(3)(5)}{6}
\end{aligned}
$$

## Savings and interest

## Example

If we deposit $\$ 2$ in a bank at the start of every year and the bank gives $3 \%$ compound interest per annum at the end of every year, how much will we have in the bank at the end of $n$ years?

Amount in bank:

| Year | Start | End |
| :---: | :--- | :--- |
| 1 | 2 | $1.03(2)$ |
| 2 | $2+1.03(2)$ | $1.03(2)+1.03^{2}(2)$ |
| 3 | $2+1.03(2)+1.03^{2}(2)$ | $1.03(2)+1.03^{2}(2)+1.03^{3}(2)$ |
| $\ldots$ | $\ldots$ | $\ldots$ |
| $n$ | $\ldots$ | $1.03(2)+1.03^{2}(2)+\ldots+1.03^{n}(2)$ |

The amount in the bank forms a geometric series with first term $1.03(2)$ and common ratio 1.03 .
$S_{n}=\frac{1.03(2)\left(1.03^{n}-1\right.}{1.1}$
Amount at end of $n$ years: $\frac{206}{3}\left(1.03^{n}-1\right)$.

## 5. Differentiation and applications

## Basics

Theory

- A GC can perform numerical differentiation
- A curve is (strictly) increasing if $f^{\prime}(x)>0$. A curve is (strictly) decreasing if $f^{\prime}(x)<0$.
- A curve is concave up if $f^{\prime \prime}(x)>0$. A curve is concave down if $f^{\prime \prime}(x)<0$.


## Tangents/normals

## Formula

- Equation of tangent/normal:

$$
y-y_{1}=m\left(x-x_{1}\right)
$$

where $m$ is the gradient of the tangent/normal at the point $\left(x_{1}, y_{1}\right)$.

- Tangents vs normals: $m_{1} m_{2}=-1$.


## Implicit differentiation

## Example

Differentiate $\left(y^{2}+5 x\right)^{5}+x^{2} y-\ln y=8$ implicitly.
$5\left(y^{2}+5 x\right)^{4}\left(2 y \frac{\mathrm{~d} y}{\mathrm{~d} x}+5\right)+2 x y+x^{2} \frac{\mathrm{~d} y}{\mathrm{~d} x}-\frac{1}{y} \frac{\mathrm{~d} y}{\mathrm{~d} x}=0$.

## Parametric equations

| Formula | Curve sketching | Example | $\frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{\cos t}{2 t}$ | Example | $\begin{aligned} & x=\frac{1}{t} \Rightarrow t=\frac{1}{x} . \\ & y=2 t^{2}-3 . \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |
|  | - A GC can sketch parametric curves. | $x=t^{2}, \quad y=\sin t$. | $y=0, \frac{\mathrm{~d} y}{\mathrm{~d} x}=-\frac{1}{2 \pi}$ | The point $\left(\frac{1}{t}, 2 t^{2}-3\right)$ | Substituting $t=\frac{1}{x}$, |
| $\frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{\mathrm{d} y}{\mathrm{~d} t} \div \frac{\mathrm{d} x}{\mathrm{~d} t}$ | - Remember to change the "tMin" and "tMax" in the " window" screen! | Find the equation of the normal when $t=\pi$. | Equation of normal: $y=2 \pi\left(x-\pi^{2}\right)$ | forms a curve as $t$ varies. Find the cartesian equation of the curve. | the cartesian equation is $y=\frac{2}{x^{2}}-3$ |

## Maxima/minima, stationary points

## Approach

- $f^{\prime}(x)=0$ at stationary points.
- For problem sums:
- Step 1: Let the quantity to be maximized/minimized be $A$, for example. Find a formula for $A$ involving other variables.
- Step 2: If necessary, form other equations and manipulate so that the formula for $A$ is in terms of only one variable ( $x$ for example).
- Step 3: Differentiate to get $\frac{d A}{d x}$.
- Step 4: At stationary values, $\frac{\mathrm{d} A}{\mathrm{~d} x}=0$. Solve for $x$.
- Step 5: Answer the question by finding the required quantities.
- Step 6: Prove that $A$ is maximum/minimum


## Example: first derivative test



Second derivative test

| $f^{\prime \prime}(b)$ | Conclusion |
| :---: | :---: |
| $f^{\prime \prime}(b)>0$ | minimum |
| $f^{\prime \prime}(b)<0$ | maximum |
| $f^{\prime \prime}(b)=0$ | no conclusion |

## Rates of change

## Approach

- Rate of change of $A: \frac{\mathrm{d} A}{\mathrm{~d} t}$.
- For problem sums:
- Step 1: Translate the rates given in the
problem sum.
- Step 2: Form an equation between two related variables ( $A$ and $x$, for example)
Step 3: Differentiate to get $\frac{\mathrm{d} A}{\mathrm{~d} x}$.
- Step 4: Apply the chain rule expression
$\frac{\mathrm{d} A}{\mathrm{~d} t}=\frac{\mathrm{d} A}{\mathrm{~d} x} \cdot \frac{\mathrm{~d} x}{\mathrm{~d} t}$
Step 5: Answer the question by finding the required quantities.


## 6. Maclaurin series

In MF26
Formula

| 1 | $f(x)=f(0)+x f^{\prime}(0)+\frac{x^{2}}{2!} f^{\prime \prime}(0)+\ldots+\frac{x^{n}}{n!} f^{(n)}(0)+\ldots$ |  |
| :--- | :--- | ---: |
| 2 | $(1+x)^{n}=1+n x+\frac{n(n-1)}{2!} x^{2}+\ldots+\frac{n(n-1) \ldots(n-r+1)}{r!} x^{r}+\ldots$ | $(\|x\|<1)$ |
| 3 | $\mathrm{e}^{x}=1+x+\frac{x^{2}}{2!}+\ldots+\frac{x^{n}}{n!}+\ldots$ | (all $x)$ |
| 4 | $\sin x=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\ldots+\frac{(-1)^{r} x^{2 r+1}}{(2 r+1)!}+\ldots$ | (all $x)$ |
| 5 | $\cos x=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\ldots+\frac{(-1)^{r} x^{2 r}}{(2 r)!}+\ldots$ | (all $x)$ |
| 6 | $\ln (1+x)=x-\frac{x^{2}}{2}+\frac{x^{3}}{3!}-\ldots+\frac{(-1)^{r+1} x^{r}}{r}+\ldots$ | $(-1<x \leq 1)$ |

## Maclaurin series using differentiation

## Approach

- To get the Maclaurin series up until and including the term in $x^{n}$, differentiate $n$ times.
- Implicit differentiation is often very useful.
- Sub in $x=0$ to obtain $y, \frac{\mathrm{~d} y}{\mathrm{~d} x}, \ldots, \frac{\mathrm{~d}^{n} y}{\mathrm{~d} x^{n}}$.
- Use formula 1 to obtain the Maclaurin series.


## Approximations using Maclaurin series

When $x$ is "small", we can often omit large powers of $x$ in the Maclaurin series and still arrive at a reasonably accurate approximation.

## Standard series

- Formulas 2-6 are often referred to as standard series.
- Start from "inside" and work outwards.


## Example

- Expand $\ln (\cos x)$ up to and including the term in $x^{4}$

$$
\ln (\cos x) \approx \ln \left(1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}\right)
$$

$$
\approx\left(-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}\right)-\frac{1}{2}\left(-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}\right)^{2}
$$

$$
\approx-\frac{x^{2}}{2}+\frac{x^{4}}{24}-\frac{1}{2}\left(\frac{x^{4}}{4}\right)
$$

$$
\approx-\frac{x^{2}}{2}-\frac{x^{4}}{12}
$$

## Small angle approximations

Formula

- $\sin x \approx x$
- $\cos x \approx 1-\frac{x^{2}}{2}$
- $\tan x \approx x$

The small angle approximations for sine and cosine can be obtained by using formulas 4 and 5 up to and including the term in $x^{2}$.

## Binomial expansion

## Theory

- Formula 6 is often referred to as the binomial expansion.
- $|x|<1$ is the range of validity of the expansion: if $x$ is within the range of validity, then the Maclaurin series converge to $(1+x)^{n}$ as $r \rightarrow \infty$.


## Example

Find the first three terms in the series expansion of $\frac{1}{\sqrt{2+x}}$.
What is the range of validity?
Range of validity: $\left|\frac{x}{2}\right|<1 \Rightarrow-2<x<2$.

$$
\begin{aligned}
\frac{1}{\sqrt{2+x}} & =(2+x)^{-\frac{1}{2}} \\
& =2^{-\frac{1}{2}}\left(1+\frac{x}{2}\right)^{-\frac{1}{2}} \\
& =\frac{1}{\sqrt{2}}\left(1-\frac{1}{2}\left(\frac{x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^{2}+\ldots\right) \\
& =\frac{1}{\sqrt{2}}\left(1-\frac{x}{4}+\frac{3 x^{4}}{32}+\ldots\right)
\end{aligned}
$$

## 7. Integration techniques



## Algebraic techniques <br> Theory <br> - Long division. <br> - Partial fraction. <br> - "Forcing" terms. <br> - Completing the square.

## Example

$$
\begin{aligned}
\int \frac{x^{2}+4 x+8}{x^{2}+2 x+5} \mathrm{~d} x & =\int 1+\frac{2 x+3}{x^{2}+2 x+5} \mathrm{~d} x \\
& =\int 1+\frac{2 x+2}{x^{2}+2 x+5}+\frac{1}{x^{2}+2 x+5} \mathrm{~d} x \\
& =\int 1+\frac{2 x+2}{x^{2}+2 x+5}+\frac{1}{(x+1)^{2}+2^{2}} \mathrm{~d} x \\
& =x+\ln \left(x^{2}+2 x+5\right)+\frac{1}{2} \tan ^{-1}\left(\frac{x+1}{2}\right)+C
\end{aligned}
$$

## Other formulas in MF26

$\int \tan x \mathrm{~d} x=\ln |\sec x|, \int \sec x \mathrm{~d} x=\ln |\sec x+\tan x|$. Formulas for $\cot x$ and $\operatorname{cosec} x$ are also provided.

## Trigonometric techniques

## Formula

$$
\begin{aligned}
& \int \sin ^{2} x \mathrm{~d} x=\int \frac{1+\cos 2 x}{2} \mathrm{~d} x \\
& \begin{aligned}
\int \sqrt{1+\cos x} \mathrm{~d} x & =\int \sqrt{1+2 \cos ^{2} \frac{x}{2}-1} \mathrm{~d} x \\
& =\int \sqrt{2} \cos \frac{x}{2} \mathrm{~d} x
\end{aligned} \\
& \int \tan ^{2} x \mathrm{~d} x=\int \sec ^{2} x-1 \mathrm{~d} x
\end{aligned} \begin{aligned}
& \int \sin 5 x \cos 3 x \mathrm{~d} x=\int \frac{1}{2}(\sin 8 x+\sin 2 x) \mathrm{d} x
\end{aligned}
$$

Integration by parts
$\int u \frac{\mathrm{~d} v}{\mathrm{~d} x} \mathrm{~d} x=u v-\int \frac{\mathrm{d} u}{\mathrm{~d} x} v \mathrm{~d} x \quad \int x \cos x \mathrm{~d} x=x \sin x-\int 1 \cdot \cos x \mathrm{~d} x$.

## 'LIATE" heuristic

- LI: We typically differentiate (i.e. let them be " $u$ ") logarithms and inverse trigonometric functions.
- TE: We typically integrate (i.e. let them be " $\frac{d v}{\mathrm{~d} x}$ ") trigonometric functions and exponential functions.
- A: Algebraic terms typically depend on who they are paired with.

```
Example: integration by parts \(\int(\sin x) \mathrm{e}^{x} \mathrm{~d} x=(\sin x) \mathrm{e}^{x}-\int(\cos x) \mathrm{e}^{x} \mathrm{~d} x\) \(=(\sin x) \mathrm{e}^{x}-\left((\cos x) \mathrm{e}^{x}-\int(-\sin x) \mathrm{e}^{x} \mathrm{~d} x\right)\)
\[
=(\sin x) \mathrm{e}^{x}-(\cos x) \mathrm{e}^{x}-\int(\sin x) \mathrm{e}^{x} \mathrm{~d} x
\]
\[
2 \int(\sin x) \mathrm{e}^{x} \mathrm{~d} x=(\sin x) \mathrm{e}^{x}-(\cos x) \mathrm{e}^{x}+C
\]
\[
\int(\sin x) \mathrm{e}^{x} \mathrm{~d} x=\frac{1}{2}\left((\sin x) \mathrm{e}^{x}-(\cos x) \mathrm{e}^{x}\right)+C
\]
```


## Integration by substitution

Approach

- Step 1: Differentiate given substitution.
- Step 2: Replace given variables ( $x$ and $\mathrm{d} x$ ).
- Step 3: Integrate.
- Step 4: Change back to the original variable.

$$
\begin{array}{rlrl}
\text { Example } & \int \frac{x}{1-x^{4}} \mathrm{~d} x & =\int \frac{x}{1-u^{2}} \cdot \frac{1}{2 x} \mathrm{~d} u \\
& \text { Use } u=x^{2} & & \frac{1}{2} \int \frac{1}{1-u^{2}} \mathrm{~d} u \\
\text { to } \text { find } & & =\frac{1}{4} \ln \left|\frac{1+u}{1-u}\right|+C \\
\int \frac{x}{1-x^{4}} \mathrm{~d} x . & & =\frac{1}{4} \ln \left|\frac{1+x^{2}}{1-x^{2}}\right|+C
\end{array}
$$

## 8. Definite integrals

## The modulus

## Approach

- For $|f(x)|$, figure out when the region when $f(x)$ is positive/negative (by drawing graphs or otherwise).
- Split up the integral. $|f(x)|=f(x)$ for regions where $f(x)$ is positive, and $|f(x)|=-f(x)$ otherwise.


## Example

Evaluate $\int_{0}^{5}\left|x^{2}-4\right| \mathrm{d} x$

$$
\begin{aligned}
\int_{0}^{5}\left|x^{2}-4\right| \mathrm{d} x & =\int_{0}^{2}-\left(x^{2}-4\right) \mathrm{d} x+\int_{2}^{5}\left(x^{2}-4\right) \mathrm{d} x \\
& =\left[-\frac{x^{3}}{3}+4 x\right]_{0}^{2}+\left[\frac{x^{3}}{3}-4 x\right]_{2}^{5} \\
& =\frac{97}{3}
\end{aligned}
$$

## Area under curves with parametric equations

Approach

- For area with respect to $x$-axis, we have $\int_{x_{1}}^{x_{2}} y \mathrm{~d} x$
- For area with respect to $y$-axis, we have $\int_{y_{1}}^{y_{2}} x \mathrm{~d} y$
- Similar to integration by substitution, we
- Differentiate to get $\frac{\mathrm{d} x}{\mathrm{~d} t}$ or $\frac{\mathrm{d} y}{\mathrm{~d} t}$ to replace $\mathrm{d} x$ or $\mathrm{d} y$.

Substitute $x$ or $y$ in terms of $t$.
Change the limits from $x$ or $y$ values to $t$ values

- Evaluate the integral.


## Volumes

Formula

- Rotation about $x$-axis:

$$
\pi \int_{x_{1}}^{x_{2}} y^{2} \mathrm{~d} x
$$

- Rotation about $y$-axis:

$$
\pi \int_{y_{1}}^{y_{2}} x^{2} \mathrm{~d} y
$$

## Examples

## Parametric equations

A curve $C$ is defined by $x=t^{2}+t, y=t-t^{3}$ for $t \geq 0$.

Find the area bounded by $C$ and the $x$-axis.
$x=t^{2}+t \Rightarrow \frac{\mathrm{~d} x}{\mathrm{~d} t}=2 t+1$.
When $y=0$,
$t=0, x=0$ or $t=1, x=2$.
Area $=\int_{0}^{2} y \mathrm{~d} x$
$=\int_{0}^{1}\left(t-t^{3}\right)(2 t+1) \mathrm{d} t=\frac{31}{60}$

## Areas and volumes 1

The region $R$ is bounded by the curve $y=x^{2}$, the line $x=2$ and the $x$-axis.

Find the area of $R$, and the volumes when $R$ is rotated about the $x$ - and $y$-axes.

Area of $R=\int_{0}^{2} x^{2} \mathrm{~d} x$
Volume when $R$ rotated about the $x$-axis $=\pi \int_{0}^{2}\left(x^{2}\right)^{2} \mathrm{~d} x$
Volume when $R$ rotated about the $y$-axis
$=$ cylinder $-\pi \int_{0}^{4} y \mathrm{~d} y$

## Areas and volumes 2

The region $S$ is bounded by the curve $y=x^{2}$, the line $y=4$ and the $y$-axis.

Find the area of $S$, and the volumes when $S$ is rotated about the $x$ - and $y$-axes.

$$
\begin{aligned}
& \text { Handling limits, Riemann sums } \\
& \text { • } \int_{a}^{b} f(x) \mathrm{d} x=-\int_{b}^{a} f(x) \mathrm{d} x \text {. } \\
& \text { • } \int_{a}^{b} u \frac{\mathrm{~d} v}{\mathrm{~d} x} \mathrm{~d} x=[u v]_{a}^{b}-\int_{a}^{b} \frac{\mathrm{~d} u}{\mathrm{~d} x} v \mathrm{~d} x . \\
& \text { - Remember to change limits when applying } \\
& \text { integration by substitution. } \\
& \int_{0}^{1} f(x) \mathrm{d} x=\lim _{n \rightarrow \infty} \frac{1}{n}\left(f\left(\frac{0}{n}\right)+f\left(\frac{1}{n}\right)+\ldots+f\left(\frac{n-1}{n}\right)\right)
\end{aligned}
$$

## Examples (continued)

Area of $S=\int_{0}^{4} \sqrt{y} \mathrm{~d} y$
Volume when $S$ rotated about the $x$-axis $=$ cylinder $-\pi \int_{0}^{2}\left(x^{2}\right)^{2} \mathrm{~d} x$
Volume when $S$ rotated about the $y$-axis $=\pi \int_{0}^{4} y \mathrm{~d} y$

## 9. Differential equations

## Theory

## Theory

- Differential equations are equations involving variables (e.g. $x, y$ ) and their derivatives (e.g. $\frac{\mathrm{d} y}{\mathrm{~d} x}$ )
- They can be solved (i.e. finding an equation between $x$ and $y$ only) by integration.
- A differential equation is of variable separable form if it can be written as

$$
g(y) \frac{\mathrm{d} y}{\mathrm{~d} x}=f(x)
$$

- Variable separable equations can be solved by proceeding to integrate:

$$
\int g(y) \mathrm{d} y=\int f(x) \mathrm{d} x
$$

- Differential equations of the form $\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}=f(x)$ can be solved by integrating twice.
- The general solution refers to all possible solutions of a differential equation. There are infinite number of solutions since the integration constant $C$ can take any value.
- A particular solution refers to one solution of a differential equation. Typically a set of values will be given to us to find $C$ in order to obtain a particular solution.


## Problem sums: rates of change <br> Rate of change of $x: \frac{\mathrm{d} x}{\mathrm{~d} t}$ <br> $$
\frac{\mathrm{d} x}{\mathrm{~d} t}=\text { rate of increase of } x-\text { rate of decrease of } x
$$

## Second order DE

$\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}=2 x$
$\frac{\mathrm{d} y}{\mathrm{~d} x}=x^{2}+C$
$y=\frac{x^{3}}{3}+C x+D$.

## Substitution

## Approach

- Step 1: Differentiate given substitution implicitly.
- Step 2: Using the given substitution and the differentials in step 1, replace the old variable (e.g. y) to the new one (e.g. $u$ ).
- Step 3: The DE in the new variable should be of separable form. Integrate it.
- Step 4: Substitute back the old variable.


## Variable separable form

$\frac{\mathrm{d} y}{\mathrm{~d} x}=x y^{2}$
$\frac{1}{y^{2}} \frac{\mathrm{~d} y}{\mathrm{~d} x}=x$
$\int \frac{1}{y^{2}} \mathrm{~d} y=\int x \mathrm{~d} x$
General solution: $-\frac{1}{y}=\frac{x^{2}}{2}+C$.
If $y=1$ when $x=0, C=-1$
Particular solution: $-\frac{1}{y}=\frac{x^{2}}{2}-1$.

## Example: substitution method

## Example

Use the substitution $u=x y$ to solve

$$
x^{2} y \frac{\mathrm{~d} y}{\mathrm{~d} x}+x y^{2}=1
$$

Differentiating $u=x y$ implicitly with respect to $x$
$\frac{\mathrm{d} u}{\mathrm{~d} x}=x \frac{\mathrm{~d} y}{\mathrm{~d} x}+y$
Hence $x \frac{\mathrm{~d} y}{\mathrm{~d} x}=\frac{\mathrm{d} u}{\mathrm{~d} x}-y$
Substituting into given DE ,
$x y\left(\frac{\mathrm{~d} u}{\mathrm{~d} x}-y\right)+u y=1$
$u \frac{\mathrm{~d} u}{\mathrm{~d} x}=1$
$\frac{u^{2}}{2}=x+C$
$u^{2}=2 x+C^{\prime}$
Solution: $(x y)^{2}=2 x+C^{\prime}$

## 10a. Vectors I

## Basics

Theory

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if $\mathbf{a}=k \mathbf{b}$ for some $k \neq 0$

- The position vector of a point $A$ with coordinates $(2,3,-4)$ is the vector $\overrightarrow{O A}=2 \mathbf{i}+3 \mathbf{j}+-4 \mathbf{j}$.
- $\overrightarrow{A B}=\overrightarrow{O B}-\overrightarrow{O A}$.
- Three points $A, B$ and $C$ are collinear if $\overrightarrow{A B}=k \overrightarrow{B C}$ for some $k \neq 0$.
- The magnitude of a vector is given by $|a \mathbf{i}+b \mathbf{j}+c \mathbf{j}|=\sqrt{a^{2}+b^{2}+c^{2}}$
- A vector $\mathbf{b}$ is a unit vector if $|\mathbf{b}|=1$.

A unit vector parallel to vector $\mathbf{a}$, denoted by $\hat{\mathbf{a}}$ can be calculated by $\hat{\mathbf{a}}=\frac{\mathbf{a}}{|\mathbf{a}|}$

- If $C$ is between $A$ and $B$ such that $A C: C B=\lambda: \mu$, the ratio theorem gives us

$$
\overrightarrow{O C}=\frac{\lambda \overrightarrow{O B}+\mu \overrightarrow{O A}}{\lambda+\mu}
$$

## Dot and cross products

Theory

- The dot/scalar product: | $\left(\begin{array}{l}a \\ b \\ c\end{array}\right) \cdot\left(\begin{array}{l}d \\ e \\ f\end{array}\right)=a d+b e+c f$ |
| :---: |
- Let $\theta$ be the angle between $\mathbf{a}$ and $\mathbf{b} . \quad \mathbf{a} \cdot \mathbf{b}=|\mathbf{a}||\mathbf{b}| \cos \theta$
- If $\mathbf{a} \cdot \mathbf{b}=0$, then $\mathbf{a}=\mathbf{0}, \mathbf{b}=\mathbf{0}$ or $\mathbf{a}$ is perpendicular to $\mathbf{b}$.
- The cross/vector product: $\left(\begin{array}{l}a \\ b \\ c\end{array}\right) \times\left(\begin{array}{l}d \\ e \\ f\end{array}\right)=\left(\begin{array}{c}b f-c e \\ -(a f-c d)) \\ a e-b d\end{array}\right)$
- Let $\theta$ be the angle between $\mathbf{a}$ and $\mathbf{b} .|\mathbf{a} \times \mathbf{b}|=|\mathbf{a}||\mathbf{b}| \sin \theta$
- If $\mathbf{a} \times \mathbf{b}=\mathbf{0}$, then $\mathbf{a}=\mathbf{0}, \mathbf{b}=\mathbf{0}$ or $\mathbf{a}$ is parallel to $\mathbf{b}$.
- If $\mathbf{n}=\mathbf{a} \times \mathbf{b}$, then $\mathbf{n}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. We call $\mathbf{n}$ the normal vector.
- Area of triangle $A B C: \left.\left|\frac{1}{2}\right| \overrightarrow{A B} \times \overrightarrow{A C} \right\rvert\,$


## Vector projection



- Length of projection of $\mathbf{a}$ onto $\mathbf{b} \quad O F=\frac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{b}|}$.
- Projection vector $\overrightarrow{O F}=\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^{2}}\right) \mathbf{b}$
- Perpendicular length from $A$ to $O B: \frac{|\mathbf{a} \times \mathbf{b}|}{\mathbf{b}}$

$$
\begin{array}{lc}
\text { Vector algebra } & \\
\qquad \begin{array}{cc}
k(\mathbf{a} \cdot \mathbf{b})=(k \mathbf{a}) \cdot \mathbf{b}=\mathbf{a} \cdot(k \mathbf{b}) & k(\mathbf{a} \times \mathbf{b})=(k \mathbf{a}) \times \mathbf{b}=\mathbf{a} \times(k \mathbf{b}) \\
\mathbf{a} \cdot \mathbf{b}=\mathbf{b} \cdot \mathbf{a} & \mathbf{a} \times \mathbf{b}=-\mathbf{b} \times \mathbf{a} \\
\mathbf{a} \cdot \mathbf{a}=|\mathbf{a}|^{2} & \mathbf{a} \times \mathbf{a}=\mathbf{0} \\
\mathbf{a} \cdot(\mathbf{b}+\mathbf{c})=\mathbf{a} \cdot \mathbf{b}+\mathbf{a} \cdot \mathbf{b} & \mathbf{a} \times(\mathbf{b}+\mathbf{c})=\mathbf{a} \times \mathbf{b}+\mathbf{a} \times \mathbf{b}
\end{array}
\end{array}
$$

## Direction cosines

The direction cosines of a vector $\mathbf{d}$ are the angles between $\mathbf{d}$ and the $x, y$ and $z$ axes respectively. $\alpha=\frac{d_{1}}{\sqrt{d_{1}^{2}+d_{2}^{2}+d_{3}^{2}}}, \beta=\frac{d_{2}}{\sqrt{d_{1}^{2}+d_{2}^{2}+d_{3}^{2}}}, \gamma=\frac{d_{3}}{\sqrt{d_{1}^{2}+d_{2}^{2}+d_{3}^{2}}}$. $\alpha^{2}+\beta^{2}+\gamma^{2}=1$.

## Equation of a line

## Formula

## Vector form

$l: \mathbf{r}=\mathbf{a}+\lambda \mathbf{d}, \quad \lambda \in \mathbb{R}$

## Cartesian form:

$$
l: \frac{x-a_{1}}{d_{1}}=\frac{y-a_{2}}{d_{2}}=\frac{x-a_{3}}{d_{3}}
$$

a: position vector of a point on the line
d: direction vector parallel to the line

| Equation of a plane |  |
| :--- | :--- |
| Formula | a: position vector <br> of a point on the <br> plane |
| Vector/parametric form: | $\mathbf{d}_{\mathbf{1}}, \mathbf{d}_{\mathbf{2}}:$ direction <br> vectors parallel to |
| $p: \mathbf{r}=\mathbf{a}+\lambda \mathbf{d}_{\mathbf{1}}+\mu \mathbf{d}_{\mathbf{2}}, \lambda, \mu \in \mathbb{R}$ | the plane |
| Scalar product form: $p: \mathbf{r} \cdot \mathbf{n}=K$ | $\mathbf{n}:$ normal vector <br> perpendicular to |
| Cartesian form: | the plane <br> $K=\mathbf{a} \cdot \mathbf{n}$ |
| $p: n_{1} x+n_{2} y+n_{3} z=K$ |  |

## 10b. Vectors II



## Example

$l_{1}: \mathbf{r}=\left(\begin{array}{c}1 \\ 2 \\ -3\end{array}\right)+\lambda\left(\begin{array}{c}1 \\ -3 \\ 2\end{array}\right), \lambda \in \mathbb{R} \quad \begin{aligned} & A(1,2,-3) \\ & B(3,-6,-2)\end{aligned}$
$l_{2}: \mathbf{r}=\left(\begin{array}{c}2 \\ -3 \\ 2\end{array}\right)+\mu\left(\begin{array}{c}-1 \\ 1 \\ 1\end{array}\right), \mu \in \mathbb{R} \quad p_{2}: 5 x+3 y+2 z=0$
$p_{1}: \mathbf{r} \cdot\left(\begin{array}{l}0 \\ 2 \\ 3\end{array}\right)=-5$


## Foot of perpendiculars

For foot of perpendicular of $B$ on $l_{1}$ :
Let $\overrightarrow{O F}=\left(\begin{array}{c}1+\lambda \\ 2-3 \lambda \\ -3+2 \lambda\end{array}\right)$ since $F$ lies on $l_{1}$.
$\overrightarrow{B F}=\overrightarrow{O F}-\overrightarrow{O B}$.
Since $\overrightarrow{B F} \perp \mathbf{d}, \overrightarrow{B F} \cdot \mathbf{d}=0$.
Solve for $\lambda$ to obtain $\overrightarrow{O F}$.
For foot of perpendicular of $B$ on $p_{1}$ :
Equation of line $B F: \mathbf{r}=\left(\begin{array}{c}3 \\ -6 \\ -2\end{array}\right)+\nu\left(\begin{array}{l}0 \\ 2 \\ 3\end{array}\right), \nu \in \mathbb{R}$
Since $F$ is the intersection point between $B F$ and $p$ :
$\left(\begin{array}{c}3 \\ -6+2 \nu \\ -2+3 \mu\end{array}\right) \cdot\left(\begin{array}{l}0 \\ 2 \\ 3\end{array}\right)=-5$.
Solve for $\nu$ to obtain $\overrightarrow{O F}$.
For point of reflection, $B^{\prime}$, of $B$ in $l_{1}$ or $p_{1}$,
by ratio theorem, $\overrightarrow{O F}=\frac{\overrightarrow{O B}+\overrightarrow{O B^{\prime}}}{2}$.

## Skew lines

Two lines are skew if

- they are not parallel: $\mathbf{d}_{\mathbf{1}} \neq k \mathbf{d}_{\mathbf{2}}$
- they do not intersect: solving their equations simultaneously does not yield unique solutions.


## Angles

## Formula

- Angle between $l_{1}$ and $l_{2}: \quad \mathbf{d}_{\mathbf{1}} \cdot \mathbf{d}_{\mathbf{2}}=\left|\mathbf{d}_{\mathbf{1}}\right|\left|\mathbf{d}_{\mathbf{2}}\right| \cos \theta$.
- Angle between $l$ and $p: \mathbf{d \cdot \mathbf { n } = | \mathbf { d } | | \mathbf { n } | \operatorname { s i n } \theta}$.
- Angle between $p_{1}$ and $p_{2}: \mathbf{n}_{\mathbf{1}} \cdot \mathbf{n}_{\mathbf{2}}=\left|\mathbf{n}_{\mathbf{1}}\right|\left|\mathbf{n}_{\mathbf{2}}\right| \cos \theta$.


## Intersections

For point of intersection between $l_{1}$ and $l_{2}$ :

$$
\left(\begin{array}{c}
1+\lambda \\
2-3 \lambda \\
-3+2 \lambda
\end{array}\right)=\left(\begin{array}{c}
2-\mu \\
-3+\mu \\
2+\mu
\end{array}\right)
$$

Solve for $\lambda$ and/or $\mu$.
For point of intersection between $l_{1}$ and $p_{1}$ :

$$
\begin{aligned}
& \left(\begin{array}{c}
1+\lambda \\
2-3 \lambda \\
-3+2 \lambda
\end{array}\right) \cdot\left(\begin{array}{l}
0 \\
2 \\
3
\end{array}\right)=-5 . \\
& \text { Solve for } \lambda .
\end{aligned}
$$

For line of intersection between $p_{1}$ and $p_{2}$,
we convert both planes to cartesian form:
$p_{1}: 2 y+3 z=-5$
$p_{2}: 5 x+3 y+2 z=0$.
We solve both equations simultaneously in our GC to get
$l: \mathbf{r}=\left(\begin{array}{c}1.5 \\ -2.5 \\ 0\end{array}\right)+\omega\left(\begin{array}{c}0.5 \\ -1.5 \\ 1\end{array}\right)$.

## Line and plane


(a) $\mathbf{d} \cdot \mathbf{n} \neq 0$

Intersect at 1 poin
Intersect at 1 point
(b) $\mathbf{d} \cdot \mathbf{n}=$
$\mathbf{a} \cdot \mathbf{n}=K$
Line lies in plane

(c) $\mathbf{d} \cdot \mathbf{n}=0$,
$\mathbf{a} \cdot \mathbf{n} \neq K$ Line parallel to plane

## 11. Complex numbers

## Basics, complex conjugates

Theory

- A complex number $z$ is of the form $z=x+y i$, where $x, y \in \mathbb{R}$ and $i^{2}=-1$.
- We call $x$ the real part $\operatorname{Re}(z)=x$ and $y$ the imaginary part $\operatorname{Im}(z)=y$.
- The complex conjugate of $z=x+y i$ is given by $z^{*}=x-y i$.
- $z+z^{*}=2 x=2 \operatorname{Re}(z)$
- $z-z^{*}=2 y i=2 i \operatorname{Im}(z)$
- $z z^{*}=x^{2}+y^{2}=|z|^{2}$

Example of complex division:
$\frac{1-i}{3+4 i}=\frac{1-i}{3+4 i} \cdot \frac{3-4 i}{3-4 i}$

$$
=\frac{-1-7 i}{25}
$$



## Modulus/argument I

Formula

- $r=|z|=\sqrt{x^{2}+y^{2}}$
- Let $\arg (z)=\theta$.
$\tan \theta=\frac{y}{x}$


## Comparing parts

Example

Solve $z^{2}+z z^{*}=8-4 i$.

## Let $z=x+y i$

$(x+y i)^{2}+(x+y i)(x-y i)=8-4 i$ $x^{2}+2 x y i-y^{2}+x^{2}+y^{2}=8-4 i$
$2 x^{2}+2 x y i=8-4 i$
Comparing real parts:
$2 x^{2}=8 \Rightarrow x= \pm 2$
Comparing imaginary parts:
$2 x y=-4 \Rightarrow y=\mp 1$
Hence $z=2-i$ or $z=-2+i$.

## Modulus/argument II

Let $\alpha=\tan ^{-1}\left|\frac{y}{x}\right|$
$\theta= \begin{cases}\alpha & \text { first quadrant } \\ \pi-\alpha & \text { second quadrant } \\ -(\pi-\alpha) & \text { third quadrant } \\ -\alpha & \text { fourth quadrant }\end{cases}$


The Argand diagram

## Complex number forms

## Formula

- Cartesian form: $z=x+y i$
- Polar/trigo form: $z=r(\cos \theta+i \sin \theta)$
- Euler/exp form:

$$
z=r \mathrm{e}^{i \theta}
$$

## The conjugate root theorem

## Theory

- Let $P(z)$ be a polynomial with real coefficients. The conjugate root theorem states that if $a+b i$ is a root to $P(z)=0$, then its conjugate $a-b i$ is also a root to $P(z)=0$.


## Modulus/argument III

$$
\begin{array}{cc}
|w z|=|w||z| & \arg (w z)=\arg (w)+\arg (z) \\
\left|\frac{w}{z}\right|=\frac{|w|}{|z|} & \arg \left(\frac{w}{z}\right)=\arg (w)-\arg (z) \\
\left|z^{n}\right|=|z|^{n} & \arg \left(z^{n}\right)=n \arg (z) \\
\left|z^{*}\right|=|z| & \arg \left(z^{*}\right)=-\arg (z)
\end{array}
$$

## A special example

$1+\mathrm{e}^{i 2 \theta}=\mathrm{e}^{i \theta} \mathrm{e}^{-i \theta}+\mathrm{e}^{i \theta} \mathrm{e}^{i \theta}$
$=e^{i \theta}\left(\mathrm{e}^{-i \theta}+\mathrm{e}^{i \theta}\right)$
$=\mathrm{e}^{i \theta}\left(2 \operatorname{Re}\left(\mathrm{e}^{i \theta}\right)\right)$
$=2 \cos \theta \mathrm{e}^{i \theta}$

## Example

Solve $3 z^{3}-7 z^{2}+17 z-5=0$ given that $1+2 i$ is a root.
Since all the coefficients are real, by the conjugate root theorem, $1-2 i$ is also a root.
By the factor theorem, $(z-(1+2 i))$ and $(z-(1-2 i))$ are factors of the cubic polynomial.
$(z-1-2 i)(z-1+2 i)=(z-1)^{2}-(2 i)^{2}=z^{2}-2 z+5$.

By long division or comparing coefficients, we can obtain the final factor $3 z-1$.
$3 z^{3}-7 z^{2}+17 z-5=$ $(z-(1+2 i)(z-(1-2 i))(3 z-1)$.

Hence $z=1+2 i, z=1-2 i$ or $z=\frac{1}{3}$.

Purely real/imaginary numbers

| Condition | Cartesian | Argument $(k \in \mathbb{Z})$ |
| :---: | :---: | :---: |
| real | $y=0$ | $\arg =k \pi$ |
| real and positive | $y=0, x>0$ | $\arg =2 k \pi$ |
| real and negative | $y=0, x<0$ | $\arg =(2 k+1) \pi$ |
| purely imaginary | $x=0$ | $\arg =\frac{(2 k+1) \pi}{2}$ |

## Elements of H2 A Level Mathematics

## Contents <br> 1. Equations and inequalities <br> 2. Functions <br> 3. Graphs and transformations <br> 4. Arithmetic and geometric progressions, the sigma notation <br> 5. Differentiation and applications <br> 6. Maclaurin series <br> 7. Integration techniques <br> 8. Definite integrals <br> 9. Differential equations <br> 10a. Vectors I <br> 10b. Vectors II <br> 11. Complex numbers



A diagram that got cut during edit

## Author's note



 and when to use general cases vs specific examples. Trying to fit each topic into a single page have added to the challenge, but we think the benefit of having a handy resource outweighs the downside of the format. We hope we have not erred too much on our choices.
 mathematical journey and hope you will find yours as rewarding.

Dedicated to all my teachers and students.

## Useful links

- https://www.seab.gov.sg/home/examinations/gce-a-level/: A Level syllabus and formula List MF26.
- https://www.desmos.com/: a useful online graphing calculator.
- http://www.adotb.xyz/: the author's website.


## Copyright information

Last updated February 9, 2019.
(C) Kelvin Soh.

Contact me at Kelvinsjk (at) gmail.com

