1. **[ACJC Prelims 17]**

   (a) i. The unit vector $\mathbf{d}$ makes angles of $60°$ with both the x- and y-axes, and θ with the z-axis, where $0° \le \theta \le 90°$. Show that $\mathbf{d}$ is parallel to $\mathbf{i}+\mathbf{j}+\sqrt{2}\mathbf{k}$. [3]

   ii. The line m is parallel to $\mathbf{d}$ and passes through the point with coordinates $(2,-1,0)$. Find the coordinates of the point on m that is closest to the point with coordinates $(3,2,0)$. [3]

   (b) The plane p_1 has equation $\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}+2\mathbf{k}) = 5$, and the line l has equation $\dfrac{x-a}{2} = \dfrac{y-1}{b} = -\dfrac{z}{2}$, where a and b are constants.

   i. Given that l lies on p_1, show that $b=1$. Find the value of a. [2]

   ii. The plane p_2 contains l and is perpendicular to p_1. Find the equation of p_2 in the form $\mathbf{r}\cdot\mathbf{n} = c$, where c is a constant to be determined. [3]

   iii. The variable point $P(x,y,z)$ is equidistant from p_1 and p_2. Find the cartesian equation(s) of the locus of P. [3]

2. **[AJC Prelims 17]**
   Planes Π_1 and Π_2 are defined by

$$\Pi_1 : x - 2y + 2z = 7,$$

$$\Pi_2 : \mathbf{r}\cdot\begin{pmatrix} a \\ 3 \\ -1 \end{pmatrix} = 8$$

   where a is a constant.

   (a) The point P has position vector $-2\mathbf{i}+\mathbf{j}+\mathbf{k}$. Find the position vector of F, the foot of the perpendicular from P to plane Π_1.
   Hence or otherwise find the shortest distance from P to plane Π_1. [5]

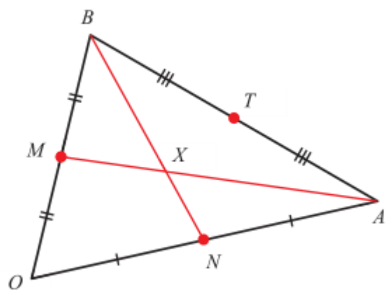
   (b) Line m passes through the point F and is parallel to both planes Π_1 and Π_2. Find a vector equation of line m. [2]

   (c) It is given that point $Q(1,-4,-1)$ lies on line m. Find the value of a. [3]

   (d) Find the length of projection of $\overrightarrow{PQ}$ on the x-y plane. [3]

3. **[CJC Prelims 17]**
   A median of a triangle is a line segment joining a vertex to the midpoint of the opposite side.

   For the triangle shown , O, A and B are vertices, where O is the origin, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The midpoints of OB, OA and AB are M, N and T respectively.

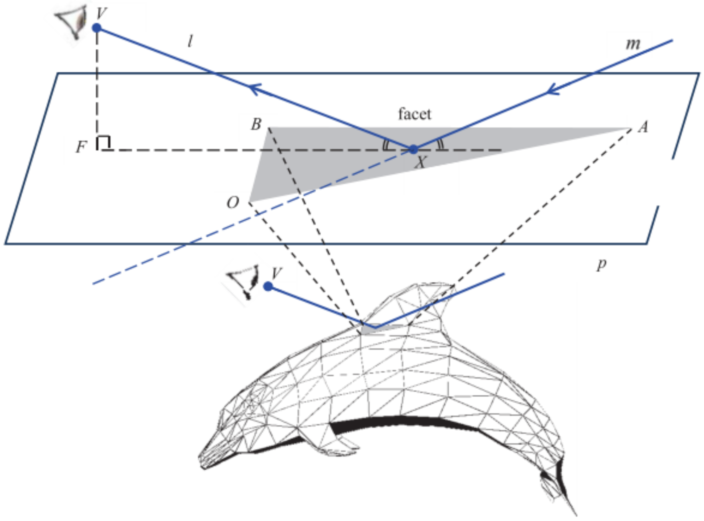


It is given that X is the point of intersection between the medians of triangle OAB from vertices A and B.

(a) Show that $\overrightarrow{OX} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$. [4]

(b) Prove that X also lies on OT, the median of triangle OAB from vertex O. [2]

The centroid of triangle OAB is the common point of intersection X between all three medians of the triangle.



Ray tracing is a technique in computer graphics rendering used to realistically capture the lighting effect in a scene being modelled. Starting from a chosen viewpoint, different rays are being traced backwards towards different parts of an object in the scene and reflected off the object. For each ray, if it reflects off the object and intersects a light source, then the part of the object at which the ray is reflected off would be made to appear brighter. In a particular scene depicting a dolphin jumping out of the ocean, a ray is being traced back from a chosen viewpoint at V to the centroid X of a particular triangular facet defined by the vertices comprising the origin O, $A(5, 4, 6)$ and $B(-2, 2, 3)$, and then reflected off the facet at X, as shown.

(c) Show that the plane p which contains the triangular facet OAB can be represented by the cartesian equation $-3y + 2z = 0$. [2]

(d) Given $V(1, -68, -37)$, determine the coordinates of the foot of the perpendicular F from V to p. [4]

The reflected ray travels along a line m such that:

- both VX (denoted by l) and line m lie in a plane that is perpendicular to plane p, and
- the angle between l and plane p equals to the angle between line m and plane p.

(e) By first finding two suitable points lying on line m, or otherwise, find a cartesian equation for line m. [5]

4. **[DHS Prelims 17)]**
The line l_1 passes through the point A, whose position vector is $3\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$ and is parallel to the vector $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$. The line l_2 is given by the cartesian equation $x - 2 = \dfrac{3 - y}{2} = \dfrac{z - 5}{2}$.
The plane p_1 contains l_1 and is parallel to l_2. Another plane p_2 also contains l_1 and is perpendicular to p_1.
   (a) Find a cartesian equation of p_1. [3]
   (b) Find the distance of l_2 to p_1. [2]
   (c) Find the equation of p_2 in scalar product form. [2]

   A particle P moves along a straight line c which lies in the plane p_2 and c passes through a point $(5, \frac{1}{2}, -3)$. P hits the plane p_1 at A and rebounds to move along another straight line d in p_2. The angle between d and l_1 is the same as the angle between c and l_1.
   (d) Find the direction cosines of d. [6]
   (e) Another particle, Q, is placed at the point $(\frac{25}{2}, \frac{21}{2}, -\frac{1}{2})$. Find the shortest distance PQ as P moves along d. [3]

5. **[DHS Prelims 17]**
Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel. The point C lies on OB produced such that $3OC = 5OB$. It is given that $|\mathbf{a}| = 2|\mathbf{b}|$ and $\cos\angle AOB = -\frac{1}{4}$.
   (a) i. Show that a vector equation of the line AC is
   $$\mathbf{r} = \mathbf{a} + \lambda(3\mathbf{a} - 5\mathbf{b}),$$
   where λ is a real parameter. [2]

   The line l lies in the plane containing O, A and B.
   ii. Explain why the direction vector of l can be expresses as $s\mathbf{a} + t\mathbf{b}$, where s and t are real numbers.
   Given that l is perpendicular to AB, show that $t = 3s$.
   Given further that l passes through B, write down a vector equation of l, in a similar form as part (i). [6]
   iii. Find the position vector of the point of intersection of AC and l, in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]
   (b) Explain why for any constant k, $|(\mathbf{a}+k\mathbf{b}) \times \mathbf{b}|$ gives the area of the parallelogram with sides OA and OB. Find the area of the parallelogram, leaving your answer in terms of $|\mathbf{a}|$. [4]

6. **[HCI Prelims 17 (modified)]**
Referred to the origin O, the position vector of a point A is $-\mathbf{i}+2\mathbf{j}+4\mathbf{k}$. A plane p contains A and is parallel to the vectors $4\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}+\mathbf{k}$.

   (a) Find a cartesian equation of p. [2]

   (b) A plane has equation $x-2y+z=2$. Find a vector equation of the line l where p and q meet. [1]

   A point B lies on l such that AB is perpendicular to l.

   (c) Find the position vector of B. [3]

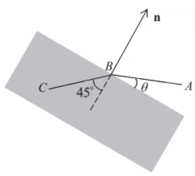
   (d) Find the length of projection of AB on the line

$$l_2 : \mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad \mu \in \mathbb{R}.$$

[2]

   (e) A point C lies on q such that AC is perpendicular to q. Find the position vector of C. Hence find a cartesian equation of the line of reflection of AB in q. [2]

7. **[IJC Prelims 17 (modified)]**
When a light ray passes from air to glass, it is deflected through an angle. The light ray ABC starts at point $A(1,2,2)$ and enters a glass object at point $B(0,0,2)$. The surface of the glass object is a plane with normal vector $\mathbf{n}$. The diagram shows a cross-section of the glass object in the plane of the light ray and $\mathbf{n}$.



   (a) Find a vector equation of the line AB. [1]

   The surface of the glass object is a plane with equation $x+z=2$. AB makes an acute angle θ with the plane.

   (b) Calculate the value of θ, giving your answer in degrees. [2]

   The line BC makes an angle of $45°$ with the normal to the plane, and BC is parallel to the unit vector $\begin{pmatrix} -\frac{2}{3} \\ p \\ q \end{pmatrix}$.

   (c) Find the values of p and q, given that $p<0$ and $q<0$. [4]

   The light ray leaves the glass object through a plane with equation $3x+3z=-4$.

   (d) Find the exact thickness of the glass object. [2]

   (e) Find the exact coordinates of the point at which the light ray leaves the glass object. [3]

8. **[TJC Prelims 17]**

   (a) The vectors **a** and **a** are the position vectors of points A and B respectively. It is given that $OA = 2\sqrt{7}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{a} \cdot \mathbf{b} = -14$.

      i. Find angle AOB. [2]

      ii. State the geometrical meaning of $|\hat{\mathbf{a}} \cdot \mathbf{b}|$, where $\hat{\mathbf{a}}$ is the unit vector of **a**. [1]

      iii. Hence or otherwise find the position vector of the foot of perpendicular from B to the line OA in terms of **a**. [2]

   (b) The non-zero vectors **p** and **q** are such that $|\mathbf{p} \times \mathbf{q}| = 2$. Given that **p** is a unit vector and $\mathbf{q} \cdot \mathbf{q} = 4$, show that **p** and **q** are perpendicular to each other. [3]

9. **[TJC Prelims 17 (modified)]**

   The point A has coordinates $(3, 1, 1)$. The line l has equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix},$$

   where λ is a parameter. P is a point on l when $\lambda = t$.

   (a) Find the cosine of the acute angle between AP and l in terms of t. Hence or otherwise find the position vector of the point N such that N is the closest point to A. [6]

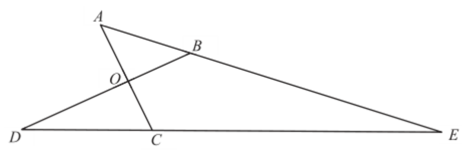
   (b) Find the coordinates of the point of reflection of A in l. [2]

   The line L has equation $x = -1, 2y = z + 2$.

   (c) Determine the relationship between L and l. [2]

   (d) Find the shortest distance from A to L. [2]

10. **[TJC Prelims 17]**



   With reference to the origin O, the points A, B, C and D are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = -\mathbf{a}$ and $\overrightarrow{OD} = -2\mathbf{b}$. The lines AB and DC meet at E.

   (a) Find $\overrightarrow{OE}$ in terms of **a** and **b**. [4]

   (b) Hence show that $\dfrac{BE}{AB} = 3$. [1]

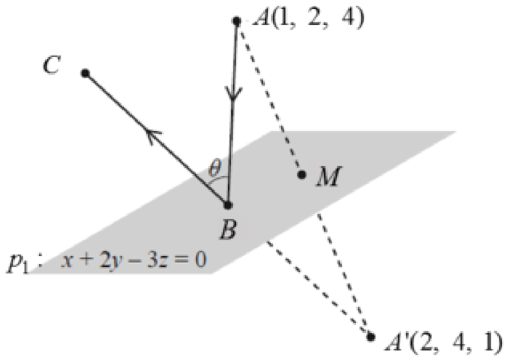
   It is given that A and E have coordinates $(1, -4, 3)$ and $(-3, 15, -5)$ respectively.

   (c) Show that the lines AC and BD are perpendicular. [4]

   (d) Find the equation of the plane p that contains E and is perpendicular to the line BD. [2]

   (e) Find the distance between the line AC and p. [2]

11. **[TPJC Prelims 17]**



Referred to the origin, a laser beam ABC is fired from the point A with coordinates $(1, 2, 4)$, and is reflected at the point B on p_1 to form a reflected ray BC as shown in the diagram. It is given that M is the midpoint of AA', where the point A' has coordinates $(2, 4, 1)$.

(a) Show that AA' is perpendicular to p_1. [2]

(b) By finding the coordinates of M, show that M lies in p_1. [2]

The vector equation of the line AB is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}.$$

(c) Find the coordinates of B.

The angle between the incident ray AB and the reflected ray BC is θ (see diagram).

(d) Given that $A'BC$ is a straight line, find the value of θ. Hence or otherwise write down the acute angle between AB and p_1. [3]

To reduce the effect of laser illumination on the pilot sitting in the cockpit at point A', a scientist proposes to include a protective film, represented by a plane p_2, such that the perpendicular distance from p_1 to p_2 is 0.5.

(e) Find the possible cartesian equations of p_2. [2]

Answers

1. (a) $(3, 0, \sqrt{2})$.
   (b) i. $a = 3$.
       ii. $\mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 4$.
       iii. $x + z = 3$.
2. (a) $-\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.
   3.
   (b) $\mathbf{r} = (-\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + \mu(- - 4\mathbf{i} + (1 + 2a)\mathbf{j} + (3 + 2a)\mathbf{k})$, where $\mu \in \mathbb{R}$.
   (c) $\frac{5}{2}$.
   (d) $\sqrt{34}$.
3. (d) $(1, -38, -57)$.
   (e) $x = 1, y - 2 = \frac{z-3}{8}$.
4. (a) $2x - y - 2z = -7$.
   (b) $\frac{2}{3}$.
   (c) $\mathbf{r} \cdot (-7\mathbf{i} + 8\mathbf{j} - 11\mathbf{k}) = 2$.
   (d) $\frac{16}{\sqrt{329}}, \frac{3}{\sqrt{329}}$ and $\frac{-8}{\sqrt{329}}$.
   (e) 2.11.
5. (a) Since the direction vector of l is co-planar with O, A and B.
   $\frac{1}{7}\mathbf{a} + \frac{10}{7}\mathbf{b}$.
   (b) $\frac{\sqrt{15}}{8}|\mathbf{a}|^2$.
6. (a) $x + y - 2z = -7$.
   (b) $\mathbf{r} = (-4\mathbf{i} - 3\mathbf{j}) + \alpha(\mathbf{i} + \mathbf{j} + \mathbf{k}), \alpha \in \mathbb{R}$.
   (c) $\mathbf{j} + 4\mathbf{k}$.
   (d) $\frac{1}{3}$.
   (e) $-\frac{1}{2}\mathbf{i} + \mathbf{j} + \frac{9}{2}\mathbf{k}$.
   $x = 0, y = 5 - z$.
7. (a) $\mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) + \lambda(\mathbf{i} + 2\mathbf{j}), \lambda \in \mathbb{R}$.
   (b) $18.4°$.
   (c) $p = -\frac{2}{3}, q = -\frac{1}{3}$.
   (d) $\frac{5\sqrt{2}}{3}$.
   (e) $(-\frac{20}{9}, -\frac{20}{9}, \frac{8}{9})$.
8. (a) $135°$.
   (b) The length of projection of $\mathbf{b}$ on $\mathbf{a}$.
   (c) $-\frac{1}{2}\mathbf{a}$.

9. (a) $3\mathbf{i} + 2\mathbf{j}$.
   (b) $(3, 3, -1)$.
   (c) They intersect at $(-1, 0, -2)$.
   (d) $\frac{9\sqrt{5}}{5}$.
10. (a) $-3\mathbf{a} + 4\mathbf{b}$.
   (d) $\mathbf{r} \cdot (3\mathbf{j} + 4\mathbf{k} = 25$.
   (e) 5.
11. (b) $(\frac{3}{2}, 3, \frac{5}{2})$.
   (c) $(0, 3, 2)$.
   (d) $80.4°$. $49.8°$.
   (e) $x + 2y - 3z = \pm\frac{\sqrt{14}}{2}$.