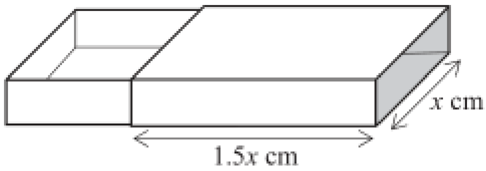
1. **[ACJC Prelims 17]**



A matchbox, shown in the diagram above, consists of an outer cover which is open at opposite ends and an inner box with an open top which slides into the outer cover. The length of the outer cover is $1.5x$ cm, where x cm is the breadth of the cover. Assume that the entire matchbox is made of cardboard with negligible thickness, and that the inner box has the same dimensions as the outer cover. If the volume of the matchbox is 30 cm^3, show that the area Acm^2 of the cardboard used to make both the inner box and outer cover is given by

$$A = 4.5x^2 + \frac{160}{x}.$$

If the amount of cardboard used is a minimum, find the exact dimensions of the matchbox, justifying that they give the minimum amount of cardboard used. [7]

2. **[ACJC Prelims 17]**

(a) The petrol consumption of a car, in millilitres per kilometre, is advertised to be $P(x) = \dfrac{1500}{x} + \dfrac{2x}{3}$ where x is the speed of the car in km/h.

i. Find the exact speed of the car when petrol consumption is minimum. [3]

ii. Sketch the graph of $y = P(x)$ for $x > 0$. [1]

iii. Hence find the range of values of x for which the petrol consumption is at most 90 millimetres per kilometre. [1]

(b) i. ** At any time t seconds, a tank is being filled with fuel at a rate given by

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 0.15\pi\sqrt{\pi t + 1},$$

where V is the volume of fuel in cm^3. Given that the tank is empty initially , find the amount of fuel in the tank after 1 minute, correct to 3 decimal places. [3]

ii. ** The tank is shaped such that when the petrol in it is at a height of h cm, the volume of petrol, V, is given by

$$V = \pi h^3.$$

Find the rate of change of h after 1 minute. [5]

3. **[AJC Prelims 17 (modified)]**
The curve C has equation $y = \mathrm{e}^{1-3x^2}$.

   (a) Without using a calculator, find the equation of the tangent to C at the point P where $x = 1$, giving your answer in the form where $y = mx + c$, where m and c are constants in exact terms to be found. [3]

   The tangent to C at P cuts the x-axis at the point A and the y-axis at the point B.

   (b) Find the exact coordinates of A and B. [2]

   (c) Find the length of AB, giving your answer to 3 significant figures. [2]

4. **[AJC Prelims 17]**
A new company manufactures souvenirs. The cost, C thousand dollars for producing x hundred souvenirs, is modelled by the equation $C = \dfrac{169}{2x+1} + 2x, 0 \leq x \leq 20$.

   (a) Use differentiation to find the number of souvenirs that must be produced to minimise the cost. State the minimum cost, justifying that this cost is a minimum. [5]

   (b) Sketch the graph of C against x, showing clearly the coordinates of any turning points and any intersections with the axes. [1]

   The daily revenue collected R thousand dollars, varies with the time t days. The CEO believes that the connection between the rate of change of the daily revenue, $\dfrac{\mathrm{d}R}{\mathrm{d}t}$, and the time t days, can be modelled by the equation $\dfrac{\mathrm{d}R}{\mathrm{d}t} = 3 - \mathrm{e}^{-2t}, t \geq 0$.

   (c) Sketch the graph of $\dfrac{\mathrm{d}R}{\mathrm{d}t}$ against t, showing clearly the coordinates of the point(s) where the curve cuts the vertical axis and the equation of any asymptote(s). Give a practical interpretation of the asymptote(s). [2]

   (d) The daily revenue collected when $t = 0$ is \$1000. Find, in terms of t, the daily revenue collected, R thousand dollars, on day t. [3]

   (e) Hence state the value of t when the daily revenue collected first reaches \$21500. [1]

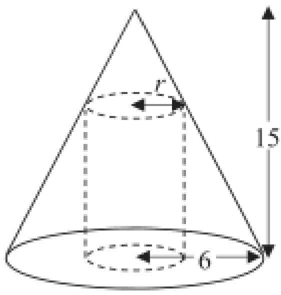
5. **[DHS Prelims 17]**
[It is given that the volume of a cone is $\frac{1}{3}\pi r^2 h$ where h is the vertical height and r the radius of the circular base of the cone.]

A hollow cone has base radius 6 cm and height 15 cm. It is made of material with negligible thickness.

(a) The cone is inverted. Initially, the cone is empty and water is poured into it at a rate of 8 cm^3s^{-1}. The depth of water in the cone is x cm at time t seconds.

i. Show that the volume V cm^3 of the water in the cone is given by $V = \frac{4}{75}\pi x^3$. [2]

ii. Find the exact rate of increase of the depth of water at the instant when the depth is 5 cm. [3]



(b) The same cone is now placed in an upright position and a solid cylinder is to be inscred in the cone (see diagram). Show that the total surface area A cm^2 of the cylinder of radius r cm is given by $A = 30\pi r - 3\pi r^2$, and hence find the exact value of the maximum A. [5]

6. **[DHS Prelims 17]**
At time 0000, a physicist observes the behaviour of 2 particles E_1 and E_2 for a period of 20 minutes. E_2 is stationary for a few minutes before it starts moving. The speed v, in m/min after t min, of E_1 and E_2 satisfies the equation $v = \frac{1}{3}t$ and $v = \sqrt{2t-5}$ respectively.

(a) On the same diagram, sketch the speed-time graphs of E_1 and E_2 during the period of observation of 20 minutes, stating the exact coordinates of any points of intersection with the axes and points of intersection of the two graphs. [2]

(b) State the duration for which E_2 moves faster than E_1. [1]

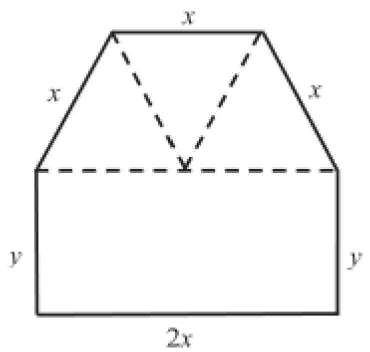
(c) ** The distance travelled is represented by the area under a speed-time graph. Determine which particle travels a longer distance for the period of observation of 20 minutes. [4]

(d) The derivative of speed with respect to time is known as acceleration. Without using a calculator, find the time at which E 1 and E 2 have the same acceleration. [3]

(e) The speed of a third particle E_3 satisfies the equation $v = \sqrt{a-(t-5)^2}$, where a is a positive constant. Find the set of values of a, given that E_1 and E_3 will not travel at the same speed at any time t. Show your working clearly. [3]

7. **[HCI Prelims 17]**
Differentiate the following with respect to x.
(a) $(x+\ln x)^2$, [2]
(b) $\mathrm{e}^{\left(\frac{1}{\sqrt{2-x}}\right)}$. [2]

8. **[HCI Prelims 17]**
(a) Kim wants to fence up a vegetable plot in his backyard. The vegetable plot to be fenced up will occupy a rectangle of $2x$ m by y m together with half of a regular hexagon with sides of x m each, as shown in the diagram below. It is given that the area of the vegetable plot is 15 m^2.



Show that the perimeter $P = 5x + \dfrac{15}{x} - \dfrac{3\sqrt{3}}{4}x$.
Find, using differentiation, the values of x and y such that P is minimum. [7]

Two companies provide the cost for the fencing.

Company A	\$90 per metre or part thereof*
Company B	\$95 per metre for the first 10 metre. \$84 for the subsequent metre of part thereof

*For example, it costs \$180 to build a fence of 1.2m using Company A.

(b) Find the range of the length of fencing to be build such that it is cheaper to engage company B. [3]
(c) Hence conclude which company Kim should engage to fence his backyard when P is minimum. [2]

9. **[IJC Prelims 17]**
Differentiate $\dfrac{3}{\sqrt{(2x-7)^3}}$. [3]

10. **[IJC Prelims 17]**

An electronic company manufactures smart phones and the manager of this company monitors how the rate of the total manufacturing costs, x million dollars per month, of their new smart phone model changes over a period of t months. The companys financial analyst believes that the relationship between x and t can be modelled by the equation

$$x = t^3 - 13t^2 + 40t + 35, \qquad \text{for } 0 \leq t \leq 12.$$

(a) Using differentiation, find the minimum value of x, justifying that this value is a minimum. [6]

(b) Sketch the graph of x against t, giving the coordinates of any intersections with the axes. [2]

(c) ** Find the area of the region bounded by the curve, the line $t = 12$ and both t- and x-axes. Give an interpretation of the area you have found, in the context of the question. [3]

The company's accountant believes that the connetion between the profit per month $\$P$ million, is related to x, by the equation

$$P = 45 + 20\ln(3x + 4)$$

(d) Find the exact value of $\frac{\mathrm{d}P}{\mathrm{d}x}$ for which $t = 8$. [2]

(e) Hence find the rate of increase in profit per month when $t = 8$. [2]

Answers

1. $x = \left(\frac{160}{9}\right)^{\frac{1}{3}}, y = (60)^{\frac{1}{3}}, h = 20\left(\frac{9}{1609}\right)^{\frac{2}{3}}$.

2. (a) $25\sqrt{6}$.
   $39.1 \leq x \leq 95.9$.

   (b) 260.755 litres.
   0.0362 cm/s.

3. (a) $y = -3\mathrm{e}^{-2}x + \frac{7}{2}\mathrm{e}^{-2}$.

   (b) $A\left(\frac{7}{6}, 0)\right), B\left(0, \frac{7}{2}\mathrm{e}^{-2})\right)$.

   (c) 1.26.

4. (a) 6000 pills, \$25000.

   (d) $R = 3t + \frac{\mathrm{e}^{-2t}}{2} + \frac{1}{2}$.

   (e) $t = 7$.

5. (a) $\frac{2}{\pi}$.

   (b) 75π.

6. (b) 12 minutes.

   (c) $E_2(69.0)$ (vs 66.7).

   (d) 00 07.

   (e) $\{a \in \mathbb{R}^+, a < \frac{5}{3}\}$.

7. (a) $2(x + \ln x)(1 + \frac{1}{x})$.

   (b) $\frac{1}{2(2-x)^{\frac{3}{2}}}\mathrm{e}^{\frac{1}{\sqrt{2-x}}}$.

8. (a) $x = 2.01, y = 2.42$.

   (b) $N \geq 19$.

   (c) Cheaper to choose Company A.

9. $\dfrac{-9}{\sqrt{(2x-7)^5}}$.

10. (a) $x = 20.2$.

    (c) 996. The total manufacturing cost to manufacture the smart phones for a period of 12 months is \$996 million.

    (d) $\frac{60}{109}$.

    (e) The rate of increase in profit when $t = 8$ is \$13.2 million per month.