The Normal Distribution

April 19, 2017

Key points

- $X \sim N(\mu, \sigma^2)$.
- $P(X = a) = 0, P(X \leq a) = P(X < a)$.
- Usage of "**normcdf**" to calculate probabilities like $P(X < 5)$, $P(X \geq 2)$ and $P(-3 \leq X < 8)$.
- Usage of "**invnorm**" to calculate a given a probability (e.g. $P(X < a) = 0.23$).
- Use of **symmetrical properties** of the normal distribution.
- If μ and/or σ is not known, **standardization** is a useful technique.

$$Z = \frac{X - \mu}{\sigma}, \qquad Z \sim N(0, 1)$$

- Linear combinations of **independent normal distributions**:

$$aX + bY$$

  - $\mathrm{E}(aX \pm bY) = a\mathrm{E}(X) \pm b\mathrm{E}(Y)$
  - $\mathrm{Var}(aX \pm bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$.

- For example, let

$$X \sim N(10, 9), \qquad Y \sim N(12, 5).$$

  - $\mathrm{E}(X - 2Y) = \mathrm{E}(X) - 2\mathrm{E}(Y) = 10 - 2(12) = -14$.
  - $\mathrm{Var}(X - 2Y) = \mathrm{Var}(X) + 2^2\mathrm{Var}(Y) = 9 + 4(5) = 29$.
  - Hence $X - 2Y \sim N(-14, 29)$.

- Understand the difference between $X_1 + X_2$ and $2X$.
- If $X \sim N(\mu, \sigma^2)$, then the **sample mean** $\overline{X} = \frac{X_1 + \cdots + X_n}{n}$ is normally distributed with

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

Past Year Questions

1. **[ACJC 12 Prelims]**
   The weekly takings, in dollars at three cinemas are modeled as independent normally distributed random variables with means and standard deviations shown in the table below.

	Mean	Standard deviation
Cinema A	6000	400
Cinema B	5100	180
Cinema C	μ	σ

   (a) Find the probability that in a randomly chosen week, the takings at cinema A will be less than cinema B. [2]
   (b) The probability that, in a randomly chosen week, the takings at cinema A exceeds x dollars is 0.975. Find x to the nearest dollar. [2]
   (c) The parent company receives a weekly levy consisting of 12% of the weekly takings at cinema A and 8% of those at cinema B. Find the probability that in a randomly chosen week, this levy exceeds \$1150. [3]
   (d) Find the probability that in exactly one week of of 5 randomly chosen week, the takings of cinema C is less than μ. [2]

2. **[ACJC 12 Prelims]**
   A machine produces sheets of steel plates of thickness X cm where $X \sim N(\mu, \sigma^2)$. Over a long period of time it has been found that 30% of the sheets are less than 29.5 cm thick and 85% of the sheets are more than 28 cm thick.

   Find, to 2 decimal places, the values of μ and σ. [5]

3. **[YJC 12 Prelims]**
   The diameters of ping-pong balls manufactured by a large factory are normally distributed with mean 4 cm and standard deviation 0.04 cm.

   (a) Find the probability that a randomly selected ping-pong ball has a diameter between 3.98 cm and 4.0 cm. [2]
   (b) Two ping-pong balls are selected at random. Find the probability that one of them has a diameter between 3.98 cm and 4.0 cm, but the other has a diameter greater than the mean. [2]
   (c) n ping-pong balls are randomly selected and placed side by side on the table. Find the greatest value of n for which there is a probability of at least 0.75 that the total diameter of the n balls is less than 68.4 cm. [4]

4. **[IJC 12 Prelims]**
The lifespan, in years, of a certain type of bulbs follows a normal distribution with mean 2 and standard deviation 0.6.

(a) Find the probability that the lifespan of a randomly chosen bulb is between 1.9 and 2.5 years. [1]

(b) Twenty bulbs are randomly taken. Find the probability that at most 3 of them have a lifespan of between 1.9 and 2.5 years. [2]

(c) * A random sample of n bulbs is taken. Find the smallest value of n such that there is a probability of at most 0.05 that the mean lifespan of these bulbs exceed 2.2 years. [5]

5. **[YJC 12 Prelims (modified)]**
The mass of sandbags of type A is normally distributed with a mean of m kg and a standard deviation of 0.25 kg.

(a) It is known that 60% of bags have mass exceeding 25 kg. Show that $m = 25.06$, correct to 2 decimal places. [3]

(b) Find the probability that one sandbag of type A weighs less than 25.2 kg. [1]

(c) Given that a sandbag exceeds 25 kg, find the probability that the sandbag weights less than 25.2 kg. [3]

(d) Find the probability that, in a random sample of 8 bags, there are fewer than 3 bags with mass exceeding 25 kg each. [2]

(e) A building company purchases 100 sandbags. Find the probability that there are fewer than 50 bags with mass less than 25 kg each. [4]

The mass of sandbags of type B is normally distributed with mean 25.05 kg and a standard deviation of 0.15 kg. The masses of sandbags of types A and B are independent.

(f) Find the probability that the total mass of 2 randomly chosen bags of type B exceeds twice the mass of a randomly chosen bag of type A by at least 0.5 kg. [4]

6. [**JJC 12 Prelims**]
The mass, in kilograms, of pears and avocados are normally distributed with means 0.17 kg and 0.15 kg respectively. The standard deviation of the two distributions are 0.024 kg and 0.012 kg respectively.

   (a) Two pears are randomly chosen. Find the probability that one pear weights more than 185 grams and the other weighs less than 185 grams. [2]

   (b) Pears are sold at $4.50 per kilogram and avocados are sold at $5.20 per kilogram. Find the probability that the total cost of 24 avocados and 12 pears is between $27 and $28. [3]

   (c) State an assumption used for the calculations in the previous part. [1]

   (d) *Find the probability that the mean weight of five randomly chosen pears is less than the mean weight of four randomly chosen avocados. [3]

7. [**AJC 10 Prelims**]
The weight of small packets of rice has a normal distribution with mean 2.5 kg and standard deviation 0.05 kg. The weight of large packets of rice has a normal distribution with mean 10 kg and standard deviation 0.15 kg.

   (a) Find the probability that four randomly chosen small packets have a total weight between 9.9 and 10.2 kg. [2]

   (b) Find the probability that one quarter of the weight of one randomly chosen large packet exceeds the weight of one randomly chosen small packet by at least 0.125 kg. [2]

   (c) A small and a large packet of rice are randomly chosen. Find the value of m if the probability of their total weight lying between 12.3 kg and m kg is 0.75. [2]

   A merchant decides to market his products in 5 kg bags. The packing process is normally distributed with a standard deviation of 0.1 kg. If the trade regulations specify that no more than 3% of the bags may be 5% under the nominal weight of 5 kg, what should the process mean weight be set at for the machinery packing the bags so that he will not violate the regulation? [4]

Numerical answers

Answers may contain mistakes. Use with care!

1. (a) 0.0201.
   (b) \$5216.
   (c) 0.330.
   (d) 0.156.
2. $\mu = 31.04, \sigma = 2.93$.
3. (a) 0.191.
   (b) 0.191.
   (c) 17.
4. (a) 0.364.
   (b) 0.0339.
   (c) 25.
5. (b) 0.712.
   (c) 0.431.
   (d) 0.0498.
   (e) 0.0160.
   (f) 0.150.
6. (a) 0.390.
   (b) 0.551.
   (c) The masses are independently distribution.
   (d) 0.0519.
7. (a) 0.819.
   (b) 0.203.
   (c) 12.7.
   4.94.