2017 H2 MA Prelim Compilation - Sigma Notation and MOD (20 Questions)

	AJC Prelim 9758/2017/01/Q4
	(a) Given that $\displaystyle\sum_{n=1}^{N}\frac{1}{4n^2-1}=\frac{1}{2}-\frac{1}{2(2N+1)}$, find $\displaystyle\sum_{n=1}^{2N}\frac{1}{4(n+1)^2-1}$. Deduce that $\displaystyle\sum_{n=1}^{2N}\frac{1}{(2n+3)^2}$ is less than $\dfrac{1}{6}$. [5] **(b)** The sum to n terms of a series is given by $S_n=n\ln 2-\dfrac{n^2-1}{\mathrm{e}}$. Find an expression for the n^{th} term of the series, in terms of n. Show that the terms of the series follow an arithmetic progression. [4]
	Answers (a) $\dfrac{1}{6}-\dfrac{1}{2(4N+3)}$ (b) $\ln 2-\dfrac{1}{e}(2n-1)$

CJC Prelim 9758/2017/01/Q3
(i) Express $\dfrac{r+1}{(r+2)!}$ in the form $\dfrac{A}{(r+1)!}+\dfrac{B}{(r+2)!}$, where A and B are integers to be found. [2] **(ii)** Find $\displaystyle\sum_{r=1}^{n}\frac{r+1}{3(r+2)!}$. [3] **(iii)** Hence, evaluate $\displaystyle\sum_{r=0}^{\infty}\frac{r+1}{3(r+2)!}$. [2]
Answers (i) $\dfrac{1}{(r+1)!}-\dfrac{1}{(r+2)!}$ (ii) $\dfrac{1}{3}\left[\dfrac{1}{2}-\dfrac{1}{(n+2)!}\right]$ (iii) $\dfrac{1}{3}$

DHS Prelim 9758/2017/01/Q1
Given that $\sum_{k=1}^{n} k!(k^2+1)=(n+1)!\,n$, find $\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)$. [3]
Answers $(n+1)!n-2$

DHS Prelim 9758/2017/01/Q2
A geometric sequence $T_1, T_2, T_3, \ldots$ has a common ratio of e. Another sequence $U_1, U_2, U_3, \ldots$ is such that $U_1=1$ and $$U_r=\ln T_r-3 \quad \text{for all } r\geq 1.$$ **(i)** Prove that the sequence $U_1, U_2, U_3, \ldots$ is arithmetic. [2] A third sequence $W_1, W_2, W_3, \ldots$ is such that $W_1=\frac{1}{2}$ and $W_{r+1}=W_r+U_r$ for all $r\geq 1$. **(ii)** By considering $\sum_{r=1}^{n-1}(W_{r+1}-W_r)$, show that $W_n=\frac{1}{2}(n^2-n+1)$. [3]
Answers --

HCI Prelim 9758/2017/02/Q1
The sum, S_n, of the first n terms of a sequence $u_1, u_2, u_3, \ldots$ is given by $$S_n=b-\frac{3a}{(n+1)!},$$ where a and b are constants. **(i)** It is given that $u_1=k$ and $u_2=\frac{2}{3}k$, where k is a constant. Find a and b in terms of k. [3] **(ii)** Find a formula for u_n in terms of k, giving your answer in its simplest form. [2] **(iii)** Determine, with a reason, if the series $\sum_{r=1}^{\infty} u_r$ converges. [1]

Answers

(i) $a=\frac{2}{3}k$, $b=2k$

(ii) $U_n=\frac{2k}{n!}\left(\frac{n}{n+1}\right)$

(iii) $S_n \to 2k$, $\sum_{r=1}^{\infty} u_r$ converges

IJC Prelim 9758/2017/01/Q5

A sequence $u_1, u_2, u_3, \ldots$ is such that

$$u_n=\frac{1}{2n^2(n-1)^2} \text{ and } u_{n+1}=u_n-\frac{2}{n(n-1)^2(n+1)^2}, \text{ for all } n \geq 2 .$$

(i) Find $\sum_{n=2}^{N} \frac{2}{n(n-1)^2(n+1)^2}$. [3]

(ii) Explain why $\sum_{n=2}^{\infty} \frac{2}{n(n-1)^2(n+1)^2}$ is a convergent series, and state the value of the sum to infinity. [2]

(iii) Using your answer in part **(i)**, find $\sum_{n=1}^{N} \frac{2N}{(n+1)n^2(n+2)^2}$. [2]

Answers

(i) $\frac{1}{8}-\frac{1}{2N^2(N+1)^2}$

(ii) $\frac{1}{8}$

(iii) $\frac{N}{8}\left[1-\frac{4}{(N+1)^2(N+2)^2}\right]$

JJC Prelim 9758/2017/01/Q3

(i) Using the formula for $\sin P - \sin Q$, show that $$\sin\left[(2r+1)\theta\right] - \sin\left[(2r-1)\theta\right] \equiv 2\cos(2r\theta)\sin\theta.$$ [1] **(ii)** Given that $\sin\theta \neq 0$, using the method of differences, show that $$\sum_{r=1}^{n}\cos(2r\theta) = \frac{\sin\left[(2n+1)\theta\right] - \sin\theta}{2\sin\theta}.$$ [2] **(iii)** Hence find $\sum_{r=1}^{n}\cos^2\left(\frac{r\pi}{5}\right)$ in terms of n. Explain why the infinite series $$\cos^2\left(\frac{\pi}{5}\right) + \cos^2\left(\frac{2\pi}{5}\right) + \cos^2\left(\frac{3\pi}{5}\right) + \ldots$$ is divergent. [3]	
Answers 3(iii) $\dfrac{\sin\dfrac{(2n+1)\pi}{5}}{4\sin\dfrac{\pi}{5}} - \dfrac{1}{4} + \dfrac{1}{2}n$	

MJC Prelim 9758/2017/01/Q5
(i) Prove by the method of differences that $$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{k}{2(n+1)(n+2)},$$ where k is a constant to be determined. [5]   **(ii)** Explain why $\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}$ is a convergent series, and state its value. [2]   **(iii)** Using your answer in part **(i)**, show that $\sum_{r=1}^{n}\frac{1}{(r+2)^3} < \frac{1}{4}$. [2]
Answers   (i) $k = 1$ (ii) $\frac{1}{4}$

NJC Prelim 9758/2017/01/Q2
(a) The sum, S_n, of the first n terms of a sequence u_1, u_2, u_3, ... is given by $S_n = 3 + 7^{-2n}\left(n^2\right)$.

(i) Write down the value of $\sum_{r=1}^{\infty} u_r$. [1]

(ii) Find a formula for u_n for $n \geq 2$ and leave it in the form $7^{-2n}\,\mathrm{g}(n)$, where $\mathrm{g}(n)$ is an expression in terms of *n*. [2]

(b) Show that $\sum_{r=1}^{n}\left(\int_0^r \mathrm{e}^{x}-\mathrm{e}^{x-1}\,\mathrm{d}x\right)=\mathrm{e}^{n}+n\mathrm{e}^{-1}-(n+1)$.

Deduce the exact value of $\sum_{r=10}^{20}\left(\int_0^r \mathrm{e}^{x+2}-\mathrm{e}^{x+1}\,\mathrm{d}x\right)$. [5]

Answers

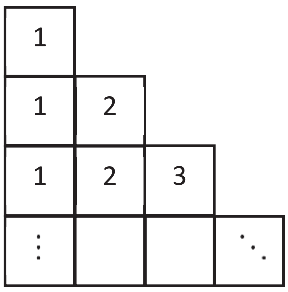
3;

$7^{-2n}(8n-7)(7-6n)$;

$\mathrm{e}^{22}-\mathrm{e}^{11}-11\mathrm{e}^{2}+11\mathrm{e}$

NYJC Prelim 9758/2017/01/Q1

A board is such that the n^{th} row from the top has n tiles, and each row is labelled from left to right in ascending order such that the i^{th} tile is labelled i, where *n* and *i* are positive integers.



Given that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, by finding the sum of the numbers in the r^{th} row, show that the sum of all the numbers in n rows of tiles is $\frac{1}{6}(n)(n+1)(n+2)$. [4]

Answers

	NYJC Prelim 9758/2017/01/Q5
	(i) By considering $f(r)-f(r+1)$, where $f(r)=\frac{\sqrt{r}}{2\sqrt{r}+1}$, find $$\sum_{r=1}^{n}\frac{\sqrt{r}-\sqrt{r+1}}{(2\sqrt{r}+1)(2\sqrt{r+1}+1)}$$ in terms of n. [3] **(ii)** Hence, find $\sum_{r=1}^{\infty}\frac{\sqrt{r}-\sqrt{r+1}}{(2\sqrt{r}+1)(2\sqrt{r+1}+1)}$. [2] **(iii)** Find the smallest integer n such that $$\sum_{r=1}^{n}\frac{\sqrt{r+1}-\sqrt{r+2}}{(2\sqrt{r+1}+1)(2\sqrt{r+2}+1)}<-0.1.$$ [3]
	Answers (i) $\frac{1}{3}-\frac{\sqrt{n+1}}{2\sqrt{n+1}+1}$ (ii) $-\frac{1}{6}$ (iii) 57

PJC Prelim 9758/2017/01/Q2
A sequence u_0, u_1, u_2, … is given by $u_0=\frac{3}{2}$ and $u_n=u_{n-1}+2^n-n$ for $n\geq 1$.   **(i)** Find u_1, u_2 and u_3. [3]   **(ii)** By considering $\sum_{r=1}^{n}(u_r-u_{r-1})$, find a formula for u_n in terms of n. [5]
Answers   (i) $u_1=\frac{5}{2}$ $u_2=\frac{9}{2}$ $u_3=\frac{19}{2}$

(ii) $u_n = 2^{n+1} - \frac{1}{2} - \frac{n(n+1)}{2}$

RI Prelim 9758/2017/02/Q2

(a) **(i)** Show that $\frac{1}{r-1} - \frac{2}{r} + \frac{1}{r+1} = \frac{2}{r(r-1)(r+1)}$. [1]

(ii) Hence find $\sum_{r=3}^{n} \frac{4}{r(r-1)(r+1)}$.

(There is no need to express your answer as a single algebraic fraction). [4]

(b) Amy and her brother Ben are saving money together for their family trip. In the first week of 2017, Amy saves \$25 and Ben saves \$2. In each subsequent week, Amy saves \$4 more than the amount she saved in the previous week, and Ben saves 22% more than the amount he saved in the previous week.

(i) Which is the first week in which Ben saves more than Amy in that week? [2]

(ii) They need a combined total of \$2400 for the trip. How many complete weeks do Amy and Ben need to save before they can achieve their targeted amount? [2]

Answers

(a)(ii) $\frac{1}{3} - \frac{2}{n} + \frac{2}{n+1}$ (b)(i) 21st week (b)(ii) 23

RVHS Prelim 9758/2017/02/Q3

(i) Express $\frac{4r+6}{(r+1)(r+2)(r+3)}$ as partial fractions. [1]

(ii) Hence find $\sum_{r=1}^{n} \frac{4r+6}{(r+1)(r+2)(r+3)}$ in terms of n. [3]

(iii) Use your answer in part **(ii)** to find the sum of the infinite series

$$\frac{3}{1\times2\times3} + \frac{5}{2\times3\times4} + \frac{7}{3\times4\times5} + \frac{9}{4\times5\times6} + \cdots.$$ [3]

Answers

(i) $\frac{4r+6}{(r+1)(r+2)(r+3)} = \frac{1}{r+1} + \frac{2}{r+2} - \frac{3}{r+3}$

	(ii) $\frac{3}{2}-\left(\frac{1}{n+2}+\frac{3}{n+3}\right)$ (iii) $\frac{5}{4}$

	SAJC Prelim 9758/2017/02/Q2
	The function f is defined by $\mathrm{f}: x \mapsto \frac{1}{x^2-1}$, $x \in \mathbb{R}$, $x>1$. **(i)** Show that $\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}=\frac{An+B}{n^3-n}$, where A and B are constants to be found. [3] **(ii)** Hence find $\sum_{r=2}^{n}\frac{2r+6}{r^3-r}$. [4] **(iii)** Use your answer to part **(ii)** to find $\sum_{r=2}^{n}\frac{2r+10}{(r+1)(r+2)(r+3)}$. [1]
	Answers i) $\frac{n+3}{n^3-n}$ ii) $3-\frac{4}{n}+\frac{2}{n+1}$ iii) $\frac{5}{6}-\frac{4}{n+2}+\frac{2}{n+3}$

	SRJC Prelim 9758/2017/02/Q1
	(i) Prove that $\frac{\sin(A-B)}{\cos A\cos B}=\tan A-\tan B$. [1] **(ii)** Hence, by considering a suitable expression of A and B, find $$\sum_{r=1}^{N}\frac{\sin x}{\cos[(r+1)x]\cos(rx)}.$$ [3] **(iii)** Using your answer to part **(ii)**, find $\sum_{r=1}^{N}\left(\frac{\sqrt{3}}{2\cos\frac{r\pi}{3}\cos\frac{(r+1)\pi}{3}}\right)$, leaving your answer in terms of N. [2]

	Answers
	(ii) $\tan(N+1)x - \tan x$ **(iii)** $\tan\left[\frac{(N+1)\pi}{3}\right] - \sqrt{3}$

TJC Prelim 9758/2017/02/Q1
Given that $\sin\left[(n+1)x\right] - \sin\left[(n-1)x\right] = 2\cos nx \sin x$, show that $$\sum_{r=1}^{n} \cos rx = \frac{\sin\left(n+\frac{1}{2}\right)x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x} \quad .$$ [4]   Hence express $$\cos^2\left(\frac{x}{2}\right) + \cos^2(x) + \cos^2\left(\frac{3x}{2}\right) + \ldots + \cos^2\left(\frac{11x}{2}\right)$$ in the form $a\left(\frac{\sin bx}{\sin cx} + d\right)$, where a, b, c and d are real numbers. [3]
Answers $\frac{1}{4}\left(\frac{\sin\frac{23}{2}x}{\sin\frac{1}{2}x} + 21\right)$

TPJC Prelim 9758/2017/01/Q7
(i) Express $\frac{1}{r^2-1}$ in partial fractions, and deduce that $$\frac{1}{r(r^2-1)} = \frac{1}{2}\left[\frac{1}{r(r-1)} - \frac{1}{r(r+1)}\right].$$ [2]   **(ii)** Hence, find the sum, S_n, of the first n terms of the series $$\frac{1}{2\times3} + \frac{1}{3\times8} + \frac{1}{4\times15} + \ldots .$$ [4]   **(iii)** Explain why the series converges, and write down the value of the sum to infinity. [2]   **(iv)** Find the smallest value of n for which S_n is smaller than the sum to infinity by less than 0.0025. [3]
Answers

(ii) $\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$ (iii) As $n \to \infty$, $\frac{1}{2(n+1)(n+2)} \to 0$. $\frac{1}{4}-\frac{1}{2(n+1)(n+2)} \to \frac{1}{4}$ Sum to infinity $=\frac{1}{4}$ (iv)13

VJC Prelim 9758/2017/01/Q8

It is given that $\sum_{r=1}^{n}\frac{r^2}{3^r}=\frac{3}{2}-\frac{n^2+3n+3}{2\left(3^n\right)}$.

(i) Find $\sum_{r=1}^{\infty}\frac{r^2+(-1)^r}{3^r}$. [3]

(ii) Show that $\sum_{r=4}^{n}\frac{(r-2)^2}{3^{r-2}}=\frac{p}{q}-\frac{an^2-an+a}{2\left(3^{n-2}\right)}$, where a, p and q are integers to be determined. [5]

Answers

(i) $\frac{5}{4}$

(ii) $p=7,\quad q=6,\quad a=1$

YJC Prelim 9758/2017/02/Q1

(i) Show that if $a_r=T_r-T_{r-1}$ for r = 1, 2, 3, …, and $T_0=0$, then

$$\sum_{r=1}^{n}a_r=T_n.$$ [1]

(ii) Deduce that $\sum_{r=1}^{n}\pi^{-r}\left[(1-\pi)r^2+2\pi r-\pi\right]=n^2\pi^{-n}$. [3]

(iii) Hence, find the exact value of $\sum_{r=4}^{20}\pi^{-r}\left[(1-\pi)r^2+2\pi r-\pi\right]$. [2]

Answers

(iii) $400\pi^{-20}-9\pi^{-3}$