Module 5: Complex Numbers (Cartesian and Polar Form)

1. **[2013/RVHS/I/7 First part]** The complex numbers w and z satisfy the equations

$$\frac{w}{z} = \frac{3}{2}\mathrm{i}$$
$$z^* - 2w = 4 + 4\mathrm{i}.$$

Find w and z. [4]

2. **[2013/DHS/II/4]**
The equation $z^4 - 4z^3 + az^2 - 20z + 25 = 0,\ a \in \mathbb{R},$ has a root ki, where k is a real number.

(i) Explain clearly why there is more than one possible value for k. [1]

(ii) Find the possible exact values of k and show that $a = 10$.
Hence find the roots of the above equation. [7]

(iii) Deduce the roots of the following equations in the form $x + \mathrm{i}y$, where x and y are real,

(a) $w^4 + 4\mathrm{i}w^3 - 10w^2 - 20\mathrm{i}w + 25 = 0,$

(b) $25v^4 - 20v^3 + 10v^2 - 4v + 1 = 0.$ [4]

3. **[2015/IJC/II/4(b)]**
The complex number w is given by $\left(\dfrac{-\sqrt{3}+\mathrm{i}}{\sqrt{2}-\mathrm{i}\sqrt{2}}\right)^2$. Without using a calculator, find

(i) $|w|$ and the exact value of $\arg w$, [4]

(ii) the set of values of n, where n is a positive integer, for which $w^n w^*$ is a real number. [4]

4. **[2015/PJC/I/7(i)]**
The polynomial P(z) has real coefficients. The equation P(z) = 0 has a root $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 < \theta \le \pi$. Write down a second root in terms of r and θ, and hence show that a quadratic factor of P(z) is $z^2 - 2rz\cos\theta + r^2$. [3]

5. **[2015/HCI/10(a)]**
The equation $z^3 - az^2 + 2az - 4\mathrm{i} = 0$, where a is a constant, has a root i.

(i) Briefly explain why i^* may not necessarily be a root of the equation. [1]

(ii) Show that $a = 2 + \mathrm{i}$. [2]

(iii) Hence, find the remaining roots of the equation in exact form. [5]

6. **[2014/MJC/I/8(b)]**
Show that $\mathrm{e}^{\mathrm{i}\frac{\theta}{2}} - \mathrm{e}^{-\mathrm{i}\frac{\theta}{2}} = 2\mathrm{i}\sin\dfrac{\theta}{2}$. Hence show that $\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}} = \dfrac{1}{2}\left(\mathrm{i}\cot\dfrac{\theta}{2} - 1\right)$. [4]

<u>Answers:</u>

1. $z=-2+2\mathrm{i},\ w=-3-3\mathrm{i}$

2. **(i)** Since the coefficients of the equation are all real, by Conjugate Root Theorem, all complex roots must occur in conjugate pairs, where $k=\pm q,\ q$ is a positive real number.

(ii) $z=\pm\sqrt{5}\mathrm{i},\ 2\pm\mathrm{i}$

(iii) $w=\pm\sqrt{5},\ \pm1-2\mathrm{i}$;
$v=\pm\dfrac{\mathrm{i}}{\sqrt{5}},\ \dfrac{1}{5}(2\pm\mathrm{i})$

3. **(ii)** $|w|=1$ and $\arg w=\dfrac{\pi}{6}$

(iii) $\{n : n\in\mathbb{Z}^+, n=6k-5,\quad k\in\mathbb{Z}^+\}$

5. **(i)** It is not necessarily true because to conclude that i^* is a root, the coefficients of the equation must be real.

(iii) $z=1+\sqrt{3}\mathrm{i}$ or $z=1-\sqrt{3}\mathrm{i}$

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