Basic Questions

1. **[Unit vectors]**

   $\mathbf{b} = (-3\mathbf{i} + \mathbf{j} + 4\mathbf{k})$. Find, in exact form,

   (a) $\hat{\mathbf{b}}$,

   (b) 2 different vectors parallel to b with length 3.

2. **[Parallel vectors/collinear points]**

   (a) The points $A(1, 0, 5)$, $B(2, 1, 4)$ and $C(x, y, 7)$ are collinear. Evaluate x and y.

   (b) Points C and D have position vectors $((\lambda - 2)\mathbf{i} + \mathbf{j} + 5\mathbf{k})$ and $(-\lambda\mathbf{i} + \lambda\mathbf{j} + \lambda 5\mathbf{k})$ respectively. Show that $\overrightarrow{CD}$ is parallel to the vector $(-2\mathbf{i} + \mathbf{j} + 5\mathbf{k})$ for all $\lambda \in \mathbb{R}, \lambda \neq 1$.

3. **[Dot and cross products]**

   (a) Prove that $(t\mathbf{i} - t\mathbf{j} + 2t\mathbf{k})$ is perpendicular to $(\mathbf{i} - \mathbf{j} - \mathbf{k})$ for all values of t.

   (b) It is given that $(t\mathbf{i} - t\mathbf{j} + 2\mathbf{k})$ is perpendicular to $(3\mathbf{i} - \mathbf{j} + \mathbf{k})$. Find the value of t.

   (c) Find, in terms of t, a vector perpendicular to both $(t\mathbf{i} - t\mathbf{j} + 2\mathbf{k})$ and $(3\mathbf{i} - \mathbf{j} + \mathbf{k})$.

4. **[Equations of lines and planes]**

   (a) Find, in all the 3 different forms (vector, parametric and cartesian) the equation of the line l passing through points $A(1, 0, 5)$ and $B(3, -2, 5)$.

   (b) Find, in all the 3 different forms (vector/parametric, scalar product form and cartesian) the equation of the plane Π passing through the point $A(2, 0, 2), B(3, -2, 5)$ and $C(1, 0, 5)$.

5. [**Distances**]

(a) Find the distance between $A(0,1,5)$ and the line

$$l\colon \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

(b) Find the distance between $B(1,0,-3)$ and the plane

$$\Pi\colon x + 2y - z = 7.$$

6. [**Intersections and angles**]

$$l_1\colon \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

$$l_2\colon \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad \mu \in \mathbb{R},$$

$$p_1\colon \mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0,$$

$$p_2\colon \mathbf{r} \cdot \begin{pmatrix} 7 \\ 1 \\ -1 \end{pmatrix} = 3.$$

(a) Find the angle between l_1 and l_2.

(b) Find the angle between l_1 and p_1.

(c) Find the coordinates of the point of intersection between l_1 and l_2.

(d) Find the position vector of the point of intersection between l_1 and p_1.

(e) Find the equation of the line of intersection between p_1 and p_2.

7. [**Foot of perpendiculars and reflections**]
(USE l_1 AND p_1 FROM QUESTION 8)
A point C has position vector $2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$.

(a) Find the position vector of the foot of perpendicular from C to l_1.

(b) Find the position vector of the foot of perpendicular from C to p_1.

(c) Hence find the position vector of C', the point obtained when C is reflected in p_1.

8. [**Relationship between lines/planes**]

$$l_1\colon \mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\3\end{pmatrix}, \quad \lambda \in \mathbb{R},$$

$$l_2\colon \mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \mu\begin{pmatrix}2\\1\\3\end{pmatrix}, \quad \mu \in \mathbb{R},$$

$$p\colon \mathbf{r}\cdot\begin{pmatrix}1\\2\\a\end{pmatrix} = b,$$

(a) Show that l_1 and l_2 are skew lines.

(b) It is given that l_1 lies in p. Find the values of a and b.

(c) It is given that l_2 and p are parallel (and non-intersecting). Find a. What can you say about b?

(d) It is given that l_2 and p intersect at exactly one point. What can you say about a?

Numerical Answers

1. $\frac{1}{\sqrt{26}}(-3\mathbf{i}+\mathbf{j}+4\mathbf{k})$.
   $\pm\frac{3}{\sqrt{26}}(-3\mathbf{i}+\mathbf{j}+4\mathbf{k})$.

2. $x=-1, y=-2$.

3. $t=\frac{1}{2}$.
   $(2-t)\mathbf{i}+(6-t)\mathbf{j}+2t\mathbf{k}$.

4. (a) $l\colon \mathbf{r}=(\mathbf{i}+5\mathbf{k})+\lambda(\mathbf{i}-\mathbf{j}), \quad \lambda\in\mathbb{R}$.
   $l\colon x=1+\lambda,\ y=-\lambda,\ z=5$.
   $l\colon x-1=-y,\ z=5$.

   (b) $\Pi\colon \mathbf{r}=(2\mathbf{i}+2\mathbf{k})+\lambda(\mathbf{i}-\mathbf{j})+\mu(\mathbf{i}-2\mathbf{j}+3\mathbf{k}), \quad \lambda,\mu\in\mathbb{R}$.
   $\Pi\colon \mathbf{r}\cdot(3\mathbf{i}+3\mathbf{j}+\mathbf{k})=8$.
   $\Pi\colon 3x+3y+z=8$.

5. (a) $\sqrt{\frac{251}{26}}$.

   (b) $\frac{3}{\sqrt{6}}$.

6. (a) 34.1°.

   (b) 34.1°.

   (c) $(1,0,-5)$.

   (d) $\mathbf{i}+\frac{1}{2}\mathbf{j}+\frac{15}{2}\mathbf{k}$.

   (e) $(\frac{2}{5}\mathbf{i}+\frac{1}{5}\mathbf{j})+\nu(\frac{1}{15}\mathbf{i}+\frac{8}{15}\mathbf{j}+\mathbf{k})$.

7. (a) $\frac{1}{26}(26\mathbf{i}-53\mathbf{j}+135\mathbf{k})$.

   (b) $\frac{1}{6}(25\mathbf{i}-44\mathbf{j}+43\mathbf{k})$.

   (c) $\frac{1}{3}(19\mathbf{i}-35\mathbf{j}+28\mathbf{k})$.

8. (b) $a=-1, b=0$.

   (c) $a=-\frac{4}{3}, b\neq-\frac{5}{3}$.

   (d) $a\neq-\frac{4}{3}$.