Vectors 2a: Lines

February 29, 2020

1. (a) Find, in both vector and cartesian form, the equation of the line l_1 passing through $A(1,0,5)$ and parallel to the vector $(2\mathbf{i}-3\mathbf{j}+5\mathbf{k})$.

   (b) Find, in both vector and cartesian form, the equation of the line l_2 passing through points $A(1,0,5)$ and $B(3,-2,5)$.

   (c) Explain whether $C(5,-6,15)$ lies in l_1.

   (d) Explain whether $C(5,-6,15)$ lies in l_2.

   (e) Find, in vector form, the equation of the line l_3 that is parallel to l_2 and passes through $C(5,-6,15)$.

   (f) * Find, in vector form, the equation of the line l_4 that passes through $A(1,0,5)$ and is perpendicular to both l_1 and l_2.

2. (a) Consider the line with equation $\dfrac{x-1}{2} = \dfrac{3-y}{3} = z$.

      i. Find a vector parallel to the line.

      ii. Find a point on the line.

      iii. Determine whether the point $(7,-3,2)$ lies on the line, giving reasons.

   (b) Find an equation of a line which passes through the point $(5,-3,2)$ and is parallel to the vector $2\mathbf{i}+\mathbf{j}-3\mathbf{k}$.

   (c) Determine whether the lines in (a) and (b) intersect. If they do intersect, find the point of intersection. If they do not, state the relationship between the lines.

   (d) Find the acute angle between the line in (a) and the line L with equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$.

3. Find the distance from the point $(2,-1,3)$ to the line

$$\frac{x-1}{2} = \frac{3+y}{3} = z$$

4. Consider the points A$(1, -1, 2)$ and B$(5, -1, -1)$.

   (a) Find the equation of the line L through A and B.

   (b) * Find the equation of the plane perpendicular to L, and which passes through A.

   (c) Find a point of L which is 20 units from A.

5. Two ships A and B have paths defined by the equations

$$\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\text{and } \begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

   respectively, where distances are in kilometers and t is the time in hours.

   (a) Find the initial position of each ship.

   (b) Find the speed of each ship.

   (c) Show that the two ships will pass through the same location, but not at the same time.

6. Suppose A is $(-1, 2, 1)$ and B is $(0, 1, 3)$.

   (a) Find the equation of the line AB in the form

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}, \lambda \in \mathrm{R}.$$

   (b) Find the angle between AB and the line L defined by

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

7. Consider the lines:

$$L_1 : x = 4 + t, y = 3 + 2t, z = -1 - 2t$$

$$L_2 : x = -1 + 3s, y = 1 - 2s, z = 2 + s$$

   (a) Classify the pair of lines as parallel, intersecting, or skew.

   (b) Find the acute angle between the lines.

8. A line, l, has equation

$$l\colon \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}, \lambda \in \mathbb{R}.$$

A point A has position vector $-\mathbf{i} + 6\mathbf{j} - 9\mathbf{k}$.

(a) Find the position vector of the foot of perpendicular from A to l.

(b) Hence find the position vector of A', the point obtained when A is reflected in l.

(c) A point B has coordinates $(1, -1, 0)$. Find the length of projection of $\overrightarrow{BA}$ on l.

Answers

1. (a) $l_1\colon \mathbf{r} = (\mathbf{i} + 5\mathbf{k}) + \lambda(2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}), \lambda \in \mathbb{R}$.
   $l_1\colon \frac{x-1}{2} = \frac{y}{-3} = \frac{z-5}{5}$.

   (b) $l_2\colon \mathbf{r} = (\mathbf{i} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j}), \mu \in \mathbb{R}$.
   $l_2\colon x - 1 = -y,\ z = 5$.

   (c) Yes, since $\lambda = 2$ when we substitute $\overrightarrow{OC}$ into the equation of the line so the system of equations is consistent.

   (d) No, since the system of equations is not consistent.

   (e) $l_3\colon \mathbf{r} = (5\mathbf{i} - 6\mathbf{j} + 15\mathbf{k}) + \nu(\mathbf{i} - \mathbf{j}), \nu \in \mathbb{R}$.

   (f) $l_4\colon \mathbf{r} = (\mathbf{i} + 5\mathbf{k}) + \omega(5\mathbf{i} + 5\mathbf{j} + \mathbf{k}), \omega \in \mathbb{R}$.

2. (a) i. $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

   ii. $(1, 3, 0)$

   iii. No

   (b) $\mathbf{r} = (5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$.

   (c) Yes, at $(5, -3, 2)$.

   (d) $40.2°$.

3. 2.31 units

4. (a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}, t \in \mathbb{R}$

   (b) $4x - 3z = -2$

   (c) $(17, -1, -10)$

5. (a) A: $(-1, 3)$, B: $(7, 4)$

   (b) A: $\sqrt{17}$ km h^{-1}, B: $\sqrt{5}$ km h^{-1}

   (c) $(3, 2)$. Ship A passes through after 1 hour, Ship B after 2 hours.

6. (a) $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$

   (b) $90°$.

7. (a) Intersect at $(2, -1, 3)$

(b) $74.5°$

8. (a) $\mathbf{i} + \mathbf{j} - 10\mathbf{k}$.
   (b) $(3, -4, -11)$.
   (c) $2\sqrt{26}$.