H

TEMASEK JUNIOR COLLEGE, SINGAPORE

JC 2
Preliminary Examination 2018
Higher 1

TEMASEK
JUNIOR COLLEGE

MATHEMATICS

8865/01

Paper 1

12 Sep 2018

Additional Materials: Answer paper

List of Formulae (MF26)

3 hours

READ THESE INSTRUCTIONS FIRST

Write your Civics group and name on all the work that you hand in.

Write in dark blue or black pen on both sides of the paper.

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Section A: Pure Mathematics [40 marks]

1 **(i)** Differentiate $\ln(\mathrm{e}+\mathrm{e}^{x})^{2}$. [2]

(ii) Given that $\int_0^a \frac{\mathrm{e}^x}{\mathrm{e}+\mathrm{e}^x}\,\mathrm{d}x = 1$, find the value of a. [4]

2 **(a)** By means of the substitution $u=\sqrt{x}$, and without the use of a graphing calculator, find the value of x which satisfies the equation $5\sqrt{x}-\frac{8}{\sqrt{x}}=6$. [3]

(b) Find, algebraically, the set of values of k for which

$$x^2-2x+kx+k^2+1>0$$

for all real values of x. [3]

3 **(i)** On the same diagram, sketch the graphs of $y=\ln(x+4)$ and $y=\frac{1}{(x-2)^2}$, showing the equations of the asymptotes and any axial intercept(s). [5]

(ii) Hence find the range of values of x for which $\ln(x+4)\geq\frac{1}{(x-2)^2}$. [3]

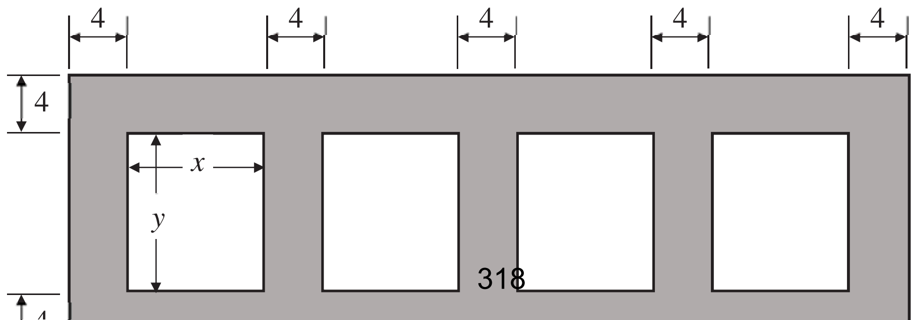
4 A particular industrial machine generates annual revenue at the rate $R'(t)=5000-20t^2$ dollars where t is the age of the machine in years. The annual operating and servicing costs for the machine is given by $C'(t)=2000+10t^2$ dollars.

(i) Sketch $y=R'(t)$ and $y=C'(t)$ on the same diagram, indicating clearly the axial intercepts. [3]

(ii) Find the range of values of t for which the machine generates a profit, that is the useful life of the machine. [2]

(iii) Use integration to find the total profit generated by the machine over the period of useful life. [3]

5



The diagram shows a photo frame design with 8 identical rectangular holes, each x cm by y cm, cut out for the display of photographs. The holes are spaced 4 cm from one another and 4 cm from the edges of the photo frame.

It is given that x and y can vary but the total area of the 8 holes must be 576 cm^2.

(i) Show that the shaded area, A cm^2, the portion of the photo frame not covered by the photos is given by

$$A = 48x + \frac{2880}{x} + 240.$$ [3]

(ii) Find the values of x and y for which A is a minimum. Hence find the minimum value of A in the form $a\sqrt{15} + b$ where a and b are integers to be found. [6]

(iii) Find the rate of change of A when x is changing at the rate of 0.2 cm s^{-1} at the instant when the value of y is twice the value of x. [3]

Section B : Statistics [60 marks]

6 A supermarket sells a particular type of durians. The masses of these durians are normally distributed with mean μ and standard deviation σ in kilograms. As part of quality control, the supermarket would discard durians that weigh less than 0.6 kg and reserve those that weigh more than 2 kg for their regular customers. Based on past data, the supermarket usually discards and reserves 15% and 1% of the durians respectively. Find the values of μ and σ. [4]

[Turn over

7

The diagram shows an observation wheel with 24 capsules. Each capsule can carry passengers up to a maximum load of 3000 kg. The weights of male passengers have mean 70 kg and standard deviation 8.9 kg.

(i) The operator of the observation wheel allows n randomly chosen male passengers to enter a capsule. Find the greatest value of n such that the probability that the total weight of the n male passengers exceed the maximum load is less than 0.01. [4]

(ii) Explain whether it is necessary to assume that the weights of male passengers are normally distributed. [1]

8 A manufacturer sells a new wifi router that is designed to have a mean signal range of 100 m. A quality control manager suspects that there is a flaw in the manufacturing process and the routers produced have a mean signal range that differs from 100 m. A random sample of 53 wifi routers is tested and found to have a mean signal range of 95.7 m and standard deviation of 11.7 m.

(i) Find an unbiased estimate of the population variance.
Explain what is meant by "unbiased estimate" in this context. [2]

(ii) Test at the 2.5% level of significance whether the quality control manager's suspicion is justified. [4]

A second sample of 53 wifi routers is tested and the unbiased estimates for the population mean and standard deviation calculated using this second sample are 97.8 m and s m respectively. A test at the 2.5% significance level does not indicate that the routers have a mean signal range of less than 100 m. Find the range of values of s that would result in such a conclusion. [4]

9 Eight students signed up for a weekly private tuition at eight different centres. They were surveyed on the monthly fees (\$$x$) they paid and their subsequent test scores (y %) after 6 months. The results are given in the following table.

Student	A	B	C	D	E	F	G	H
x	200	300	180	340	220	280	400	500
y	44	51	62	56	48	50	59	65

(i) Plot a scatter diagram for the data. Giving a reason, identify a data pair which should be regarded as suspect. [2]

The suspect data pair is subsequently removed from the data set.

(ii) Calculate the correlation coefficient for the revised data set. Comment on the value obtained. [2]

(iii) Find the equation of the regression line of y on x, and use it to predict the test score of a student who is paying \$350 for tuition, correct to the nearest integer value. Comment on the reliability of your prediction. [3]

(iv) A new equation of the regression line of y on x, $y = 32.0555 + 0.066149x$ is obtained when a new data pair was added. If the value of x of this data pair is 550, find the corresponding value of y. [3]

10 A machine is used to generate codes consisting of two integers followed by four letters. Each of the two integers generated is equally likely to be any of the nine integers 1 – 9. The integer 0 is not used. Each of the four letters generated is equally likely to be any of the seven letters of the alphabets $\{A, B, C, D, E, F, G\}$.

(i) Find the number of codes that can be formed, if no letter or integer is repeated in the code. [2]

From (ii) onwards, letters and integers can be repeated in the codes.

Find the number of codes that can be formed

(ii) with two same integers, [2]

(iii) with exactly one vowel and three consonants. [2]

Hence find the **probability** that the last letter of a randomly chosen code is a vowel given that there are exactly one vowel and three consonants. [3]

[Turn over

11 A confectionary produces a large number of sweets every day. On average, 20% of the sweets are wasabi-flavoured and the rest are caramel-flavoured.

(i) A random sample of n sweets is chosen. If the probability that there are at least three wasabi-flavoured sweets in the sample is at least 0.7, find the least possible value of n. [3]

The manufacturer decides to put the sweets randomly into packets of 20.

(ii) Find the probability that such a packet contains less than 3 wasabi-flavoured sweets. [2]

(iii) A customer selects packets of 20 sweets at random from a large consignment until she finds a packet with exactly 12 caramel-flavoured sweets. Give a reason why a Binomial Distribution is not an appropriate model for the number of packets she selects in the context of the question. [1]

The packets are then packed into boxes. Each box contains 10 packets.

(iv) Find the probability that all the packets in a randomly chosen box contain at least 3 wasabi-flavoured sweets. [2]

(v) Find the probability that there are at least 30 wasabi-flavoured sweets in a randomly chosen box. [1]

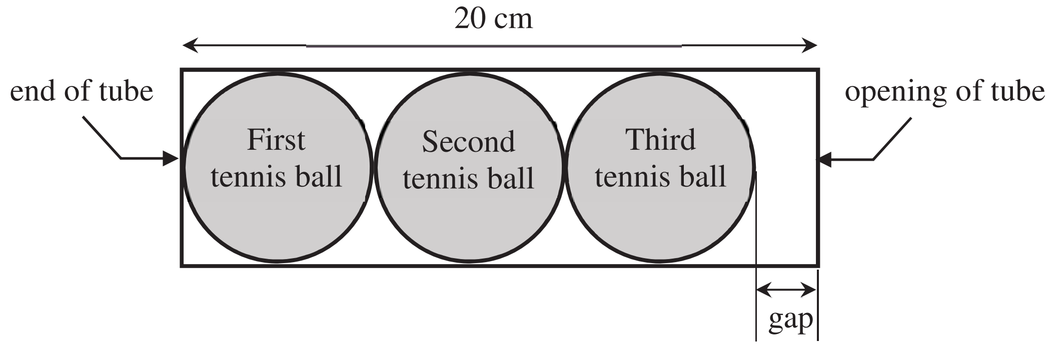
(vi) Explain why the answer to **(v)** is greater than the answer to **(iv)**. [1]

12 A company manufactures tennis balls and packs them into cylindrical tubes for sale. The tennis balls have radii that are normally distributed with mean 3.3 cm and standard deviation 0.2 cm.

(i) Find the probability that the radius of a randomly selected tennis ball lies between 3.135 cm and 3.465 cm. Without any further calculation, explain, with the aid of a diagram, how the answer obtained would compare with the probability that the radius lies between 3.465 cm and 3.795 cm. [3]

(ii) 3 tennis balls are randomly selected. Find the probability that exactly one of them has a radius less than 3.4 cm and two of them have radii greater than 3.4 cm each. [2]

The cylindrical tubes are 20 cm long. 3 tennis balls are randomly selected and packed into a cylindrical tube such that the first tennis ball is in contact with the end of the tube and each subsequent ball is in contact with its neighbouring ball as shown in the diagram below. Assume that the centres of all the tennis balls are horizontally aligned.



(iii) Find the probability that a gap exists between the third tennis ball and the opening of the tube. [3]

(iv) Find the range of values of k such that the probability that the gap between the third tennis ball and the opening of the tube is more than k cm is at most 0.15. [3]

State an assumption used in your calculations. [1]

--- End of paper ---

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Section A: Pure Mathematics [40 marks]

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(ii) Given that $\int_0^a \frac{\mathrm{e}^x}{\mathrm{e}+\mathrm{e}^x}\,\mathrm{d}x = 1$, find the value of a. [4]

[Solution]

(i) $$\frac{\mathrm{d}}{\mathrm{d}x}\ln(\mathrm{e}+\mathrm{e}^x)^2 = 2\frac{\mathrm{d}}{\mathrm{d}x}\ln(\mathrm{e}+\mathrm{e}^x)$$
$$= \frac{2\mathrm{e}^x}{\mathrm{e}+\mathrm{e}^x}$$

(ii) From (i), $\int \frac{2\mathrm{e}^x}{\mathrm{e}+\mathrm{e}^x}\,\mathrm{d}x = \ln(\mathrm{e}+\mathrm{e}^x)^2$

$$\int_0^a \frac{2\mathrm{e}^x}{\mathrm{e}+\mathrm{e}^x}\,\mathrm{d}x = \left[\ln(\mathrm{e}+\mathrm{e}^x)^2\right]_0^a$$
$$2\int_0^a \frac{\mathrm{e}^x}{\mathrm{e}+\mathrm{e}^x}\,\mathrm{d}x = 2\ln(\mathrm{e}+\mathrm{e}^a) - 2\ln(\mathrm{e}+1)$$
$$2\int_0^a \frac{\mathrm{e}^x}{\mathrm{e}+\mathrm{e}^x}\,\mathrm{d}x = 2\ln\left(\frac{\mathrm{e}+\mathrm{e}^a}{\mathrm{e}+1}\right)$$

Given $\int_0^a \frac{\mathrm{e}^x}{\mathrm{e}+\mathrm{e}^x}\,\mathrm{d}x = 1$

$$1 = \ln\left(\frac{\mathrm{e}+\mathrm{e}^a}{\mathrm{e}+1}\right)$$
$$\frac{\mathrm{e}+\mathrm{e}^a}{\mathrm{e}+1} = \mathrm{e}$$
$$\mathrm{e}+\mathrm{e}^a = \mathrm{e}^2+\mathrm{e}$$
$$a = 2$$

2 **(a)** By means of the substitution $u=\sqrt{x}$, and without the use of a graphing calculator, find the value of x which satisfies the equation $5\sqrt{x}-\dfrac{8}{\sqrt{x}}=6$. [3]

(b) Find, algebraically, the set of values of k for which

$$x^2-2x+kx+k^2+1>0$$

for all real values of x. [3]

[Solution]

(a)

$$5\sqrt{x}-\frac{8}{\sqrt{x}}=6 \Rightarrow 5u-\frac{8}{u}=6$$

$$5u^2-6u-8=0$$

$$(5u+4)(u-2)=0$$

$$u=-\frac{4}{5} \quad \text{or} \quad u=2$$

$$\sqrt{x}=-\frac{4}{5}\ \text{(Reject) or} \quad \sqrt{x}=2$$

$$x=4$$

(b) Coefficient of $x^2>0$ and Discriminant <0 since the graph is above x-axis.

$$(-2+k)^2-4\left(k^2+1\right)<0$$

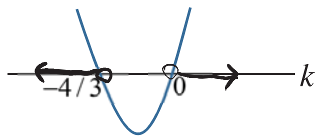
$$4-4k+k^2-4k^2-4<0$$

$$-3k^2-4k<0$$

$$3k^2+4k>0$$

$$k(3k+4)>0$$

$$k<-\frac{4}{3} \quad \text{or} \quad k>0$$



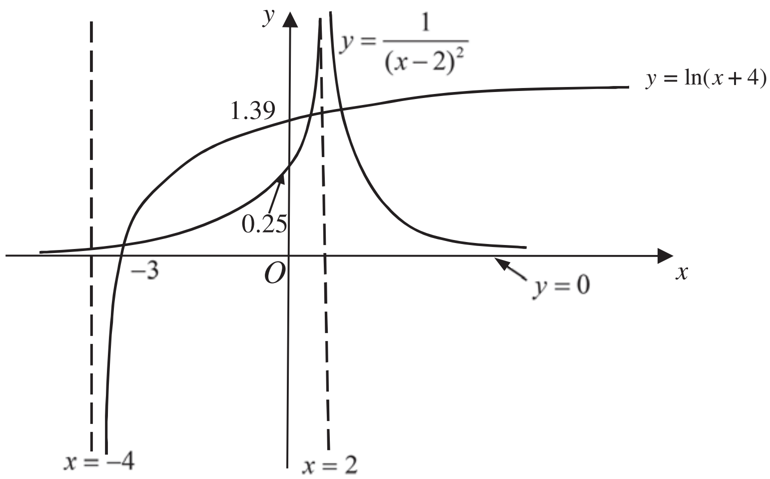
The set of values of k is $\{k\in\mathbb{R} : k<-\dfrac{4}{3} \quad \text{or} \quad k>0\}$

3 **(i)** On the same diagram, sketch the graphs of $y=\ln(x+4)$ and $y=\frac{1}{(x-2)^2}$, showing the equations of the asymptotes and any axial intercept(s). [5]

(ii) Hence find the range of values of x for which $\ln(x+4)\geq\frac{1}{(x-2)^2}$. [3]

[Solution]

(i)



(ii) The intersections are $x=-2.96$, 1.22 and 2.72

From the graph, $-2.96\leq x\leq 1.22$ or $x\geq 2.72$

4 A particular industrial machine generates annual revenue at the rate $R'(t)=5000-20t^2$ dollars where t is the age of the machine in years. The annual operating and servicing costs for the machine is given by $C'(t)=2000+10t^2$ dollars.

(i) Sketch $y = R'(t)$ and $y = C'(t)$ on the same diagram, indicating clearly the axial intercepts. [3]

(ii) Find the range of values of t for which the machine generates a profit, that is the useful life of the machine. [2]

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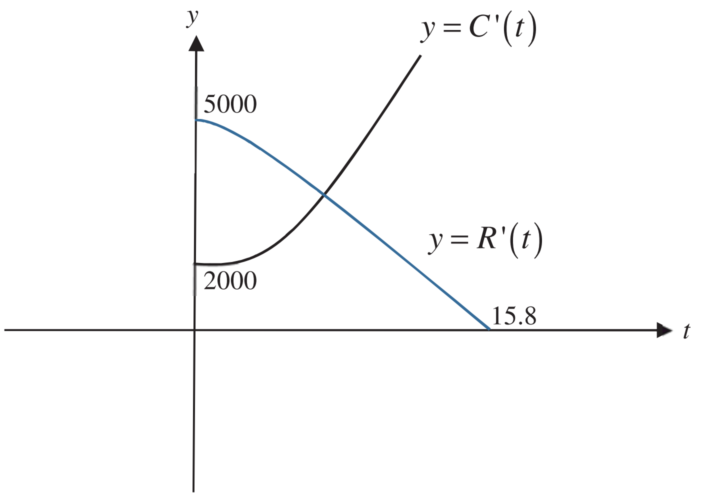
[Solution]

(i)



(ii) From GC, $t = 10$

For $R'(t) > C'(t)$, the range of values of t is $0 \le t < 10$.

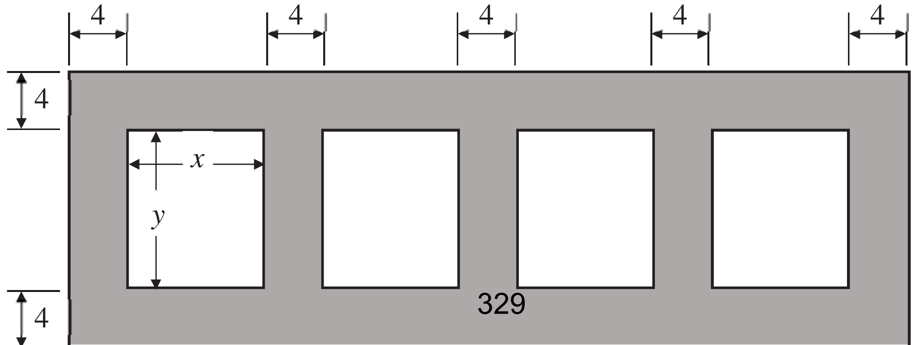
(iii) Total profit generated = $\displaystyle\int_0^{10} \left[5000 - 20t^2 - \left(2000 + 10t^2\right)\right] \mathrm{d}t$

$$= \int_0^{10} \left[3000 - 30t^2\right] \mathrm{d}t$$

$$= \left[3000t - \frac{30}{3}t^3\right]_0^{10}$$

$$= 20000$$

5



The diagram shows a photo frame design with 8 identical rectangular holes, each x cm by y cm, cut out for the display of photographs. The holes are spaced 4 cm from one another and 4 cm from the edges of the photo frame.

It is given that x and y can vary but the total area of the 8 holes must be 576 cm^2.

(i) Show that the shaded area, A cm^2, the portion of the photo frame not covered by the photos is given by

$$A = 48x + \frac{2880}{x} + 240.$$ [3]

(ii) Find the values of x and y for which A is a minimum. Hence find the minimum value of A in the form $a\sqrt{15}+b$ where a and b are integers to be found. [6]

(iii) Find the rate of change of A when x is changing at the rate of 0.2 cm s^{-1} at the instant when the value of y is twice the value of x. [3]

[Solution]

(i) Total area of 8 holes, $8xy = 576 \Rightarrow y = \frac{72}{x}$

Area of shaded region,

$$\begin{aligned} A &= (4x+5\times4)(2y+3\times4) - 8xy \\ &= 8xy + 48x + 40y + 240 - 8xy \\ &= 48x + \frac{2880}{x} + 240 \quad \text{(shown)} \end{aligned}$$

(ii) $\frac{dA}{dx} = 48 - \frac{2880}{x^2}$

$\frac{dA}{dx} = 0 \Rightarrow 48 - \frac{2880}{x^2} = 0$

$\Rightarrow \quad x = \sqrt{60}$ or $-\sqrt{60}$ (reject as $x > 0$)

$\Rightarrow \quad y = \frac{72}{\sqrt{60}}$

x	$\sqrt{60}^-$	$\sqrt{60}$	$\sqrt{60}^+$
$\frac{\mathrm{d}A}{\mathrm{d}x}$	− ve	0	+ ve

$\therefore$ A is minimum at $x=\sqrt{60}$.

Alternative: Using the second derivative test,

$\frac{\mathrm{d}^2 A}{\mathrm{d}x^2}=\frac{5600}{x^3}>0$ for $x=\sqrt{60}$

$\therefore$ A is minimum at $x=\sqrt{60}$.

$$A=48\sqrt{60}+\frac{2880}{\sqrt{60}}\times\frac{\sqrt{60}}{\sqrt{60}}+240=48\times2\sqrt{15}+48\sqrt{60}+240=96\sqrt{15}+96\sqrt{15}+240=192\sqrt{15}+240$$

Minimum A is $192\sqrt{15}+240$ where a = 192 and b = 240

(iii) $y=2x \quad \Rightarrow \quad 2x=\frac{72}{x}$

$\Rightarrow \quad x=6$

$\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$

$\frac{\mathrm{d}A}{\mathrm{d}t}=\left(48-\frac{2800}{x^2}\right)\times0.2$

$\frac{\mathrm{d}A}{\mathrm{d}t}=\left(48-\frac{2880}{6^2}\right)\times0.2$

$=-\frac{32}{5}$ (or -6.4)

A is decreasing at rate of $+\frac{32}{5}$ cm^2 s^{-1}

Section B : Statistics [60 marks]

6 A supermarket sells a particular type of durians. The masses of these durians are normally distributed with mean μ and standard deviation σ in kilograms. As part of quality control, the supermarket would discard durians that weigh less than 0.6 kg and reserve those that weigh more than 2 kg for their regular customers. Based on past data, the supermarket usually discards and reserves 15% and 1% of the durians respectively. Find the values of μ and σ. [4]

[Solution]

Let X denote the weight of a particular type of durian.

$X \sim N(\mu, \sigma^2)$

$\mathrm{P}(X < 0.6) = 0.15$

$\mathrm{P}\left(Z < \dfrac{0.6-\mu}{\sigma}\right) = 0.15$

From GC, $\dfrac{0.6-\mu}{\sigma} = -1.0364 ------(1)$

$\mathrm{P}(X > 2) = 0.01$

$\mathrm{P}\left(Z > \dfrac{2-\mu}{\sigma}\right) = 0.01$

From GC, $\dfrac{2-\mu}{\sigma} = 2.3263 ------(2)$

Solving, $\sigma = 0.416$, $\mu = 1.03$ (shown)

7

The diagram shows an observation wheel with 24 capsules. Each capsule can carry passengers up to a maximum load of 3000 kg. The weights of male passengers have mean 70 kg and standard deviation 8.9 kg.

(i) The operator of the observation wheel allows n randomly chosen male passengers to enter a capsule. Find the greatest value of n such that the probability that the total weight of the n male passengers exceed the maximum load is less than 0.01. [4]

(ii) Explain whether it is necessary to assume that the weights of male passengers are normally distributed. [1]

[Solution]

Let M be the weight of a male passenger
$\mathrm{E}(M) = 70$, $\mathrm{Var}(M) = 8.9^2$

Let $T = M_1 + M_2 + M_3 + \ldots + M_n$
Assuming that n is large, by Central Limit Theorem,
$T \sim \mathrm{N}(70n, 8.9^2 n)$ approximately

(i) $\mathrm{P}(T > 3000) < 0.01$

From GC

n	$\mathrm{P}(T > 3000)$
40	$0.00019 < 0.01$
41	$0.0113 > 0.01$

Greatest value of n is 40

(ii) It is not necessary to assume that the weights of male passengers are normally distributed as the total weight of n male passengers is approximately normally distributed by the Central Limit Theorem since n is large.

8 A manufacturer sells a new wifi router that is designed to have a mean signal range of 100 m. A quality control manager suspects that there is a flaw in the manufacturing process and the routers produced have a mean signal range that differs from 100 m. A random sample of 53 wifi routers is tested and found to have a mean signal range of 95.7 m and standard deviation of 11.7 m.

(i) Find an unbiased estimate of the population variance.
Explain what is meant by "unbiased estimate" in this context. [2]

(ii) Test at the 2.5% level of significance whether the quality control manager's suspicion is justified. [4]

A second sample of 53 wifi routers is tested and the unbiased estimates for the population mean and standard deviation calculated using this second sample are 97.8 m and *s* m respectively. A test at the 2.5% significance level does not indicate that the routers have a mean signal range of less than 100 m. Find the range of values of *s* that would result in such a conclusion. [4]

[Solution]

(i) Let X be the signal range of a router and μ be the population mean signal range.

Unbiased estimate of the population variance is $s^2 = \frac{53}{52}\left(11.7^2\right) = 139.5225$

s^2 is an unbiased estimate of the population variance means that the mean of the sampling distribution of S^2, i.e. $E(S^2)$ is equal to the actual population variance.

(ii) $H_0 : \mu = 100$

$H_1 : \mu \neq 100$

Level of significance: 2.5%

Under H_0, test statistic: $Z = \dfrac{\overline{X} - 100}{\frac{\sqrt{139.5225}}{\sqrt{53}}} \sim N(0,1)$ approximately

by Central Limit Theorem since n =53 is large

Given $\bar{x} = 95.7$

From GC, p-value $= 0.00804 < 0.025$

Since p-value < level of significance, we reject H_0.

There is sufficient evidence at 2.5% significance level that the quality control manager's suspicion that the routers produced have a mean signal range that differs from 100 m is justified.

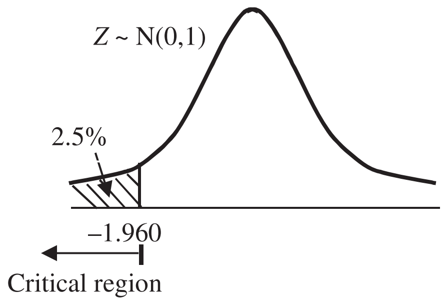
(iii) $H_0 : \mu = 100$

$H_1 : \mu < 100$

Level of significance: 2.5%

test statistic: $Z = \dfrac{\overline{X} - 100}{\frac{s}{\sqrt{53}}} \sim N(0,1)$ approximately

by Central Limit Theorem since n = 53 is large



Since H_0 is not rejected,

z_{cal} lies outside the critical region

$$\frac{97.8 - 100}{\frac{s}{\sqrt{53}}} > -1.95996$$

$\Rightarrow \quad s > 8.17 \quad$ (3 sig fig)

9 Eight students signed up for a weekly private tuition at eight different centres. They were surveyed on the monthly fees ($\$x$) they paid and their subsequent test scores (y %) after 6 months. The results are given in the following table.

Student	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
x	200	300	180	340	220	280	400	500
y	44	51	62	56	48	50	59	65

(i) Plot a scatter diagram for the data. Giving a reason, identify a data pair which should be regarded as suspect. [2]

The suspect data pair is subsequently removed from the data set.

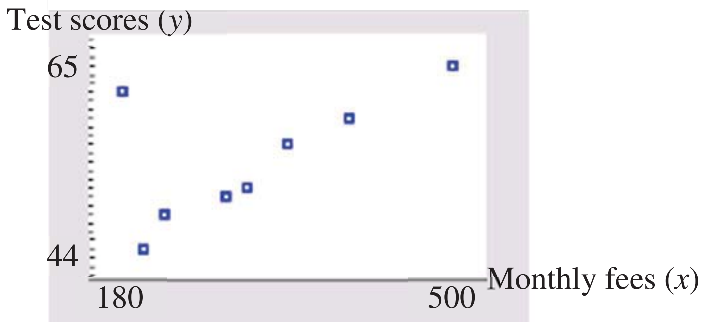
(ii) Calculate the correlation coefficient for the revised data set. Comment on the value obtained. [2]

(iii) Find the equation of the regression line of y on x, and use it to predict the test score of a student who is paying \$350 for tuition, correct to the nearest integer value. Comment on the reliability of your prediction. [3]

(iv) A new equation of the regression line of y on x, $y = 32.0555 + 0.066149x$ is obtained when a new data pair was added. If the value of x of this data pair is 550, find the corresponding value of y. [3]

[Solution]

(i)



The data pair (180, 62) should be regarded as suspect because it does not follow the trend that as x increases, y increases.

(ii) From GC, $r = 0.989$. There is a strong positive linear correlation between x and y.

(iii) $y = 31.6 + 0.0677x$ (3 s.f.)

When $x = 350$, $y = 31.627 + 0.067683(350) = 55.3 \approx 55$

Since $x = 350$ is within the data range [180, 500] and $r = 0.989$ is close to 1, the prediction is reliable.

(iv) Let the unknown y value be k.

$\bar{x} = 348.75$

$\bar{y} = \dfrac{373 + k}{8}$

Since $(\bar{x}, \bar{y})$ lies on the regression line,

$\dfrac{373 + k}{8} = 32.0555 + 0.066149(348.75)$

$\Rightarrow \quad k = 68$

10 A machine is used to generate codes consisting of two integers followed by four letters. Each of the two integers generated is equally likely to be any of the nine integers 1 – 9. The integer 0 is not used. Each of the four letters generated is equally likely to be any of the seven letters of the alphabets $\{A, B, C, D, E, F, G\}$.

(i) Find the number of codes that can be formed, if no letter or integer is repeated in the code. [2]

From (ii) onwards, letters and integers can be repeated in the codes.

Find the number of codes that can be formed

(ii) with two same integers, [2]

(iii) with exactly one vowel and three consonants. [2]

Hence find the **probability** that the last letter of a randomly chosen code is a vowel given that there are exactly one vowel and three consonants. [3]

[Solution]

(i) Number of codes required = ${}^9C_2 \times 2! \times {}^7C_4 \times 4! = 60480$

Alternative

- ${}^9P_2 \times {}^7P_4$
- $(9\times 8) \times (7\times 6\times 5\times 4)$

(ii) Number of codes required = $9\times 7^4 = 21609$

$$\underbrace{(9\times 1)}_{2^{\text{nd}}\text{ integer same}} \times \underbrace{(7\times 7\times 7\times 7)}_{\text{letters can repeat}}$$

(iii) No. of codes = $(9^2)\times(5^3\times 2)\times 4 = 81000$

Notes: 9^2 : integers can repeat
5^3 : 3 consonant letters can repeat
2^1 : 1 vowel chosen can be letter A or E
4 : 1 vowel chosen can be in any of the 4 positions CCCV, CCVC, CVCC, VCCC

$$\text{P}(\text{last letter is a vowel} \mid \text{exactly one vowel and three consonants})$$
$$= \frac{\text{P}(\text{last letter is a vowel and there are exactly one vowel and three consonants})}{\text{P}(\text{exactly one vowel and three consonants})}$$

$$= \frac{\dfrac{9^2\times 5^3\times 2}{9^2\times 7^4}}{\dfrac{81000}{9^2\times 7^4}} = 0.25$$

1 vowel chosen can only be in last position

sample space for both probabilities: total no. of codes = $9^2\times 7^4$

[M1, A1]

11 A confectionary produces a large number of sweets every day. On average, 20% of the sweets are wasabi-flavoured and the rest are caramel-flavoured.

(i) A random sample of n sweets is chosen. If the probability that there are at least three wasabi-flavoured sweets in the sample is at least 0.7, find the least possible value of n. [3]

The manufacturer decides to put the sweets randomly into packets of 20.

(ii) Find the probability that such a packet contains less than 3 wasabi-flavoured sweets. [2]

(iii) A customer selects packets of 20 sweets at random from a large consignment until she finds a packet with exactly 12 caramel-flavoured sweets. Give a reason why a Binomial Distribution is not an appropriate model for the number of packets she selects in the context of the question. [1]

The packets are then packed into boxes. Each box contains 10 packets.

(iv) Find the probability that all the packets in a randomly chosen box contain at least 3 wasabi-flavoured sweets. [2]

(v) Find the probability that there are at least 30 wasabi-flavoured sweets in a randomly chosen box. [1]

(vi) Explain why the answer to **(v)** is greater than the answer to **(iv)**. [1]

[Solution]

(i) Let X be the number of wasabi-flavoured sweets out of n.

$X \sim \mathrm{B}(n,\ 0.2)$

$\mathrm{P}(X \geq 3) \geq 0.7$

$1-\mathrm{P}(X \leq 2) \geq 0.7$

n	$1-\mathrm{P}(X \leq 2)$
17	$0.6904 < 0.7$
18	$0.7287 > 0.7$

Least n is 18.

(ii) Let Y be the number of wasabi-flavoured sweets out of 20.

$$Y \sim \mathrm{B}(20,\ 0.2)$$

$$\mathrm{P}(Y<3)=\mathrm{P}(Y\le 2)$$
$$=0.206$$

(iii) The number of packets selected (i.e, the number of trials) is not fixed.

(iv) Let W be the number of packets which contains at least 3 wasabi-flavoured sweets out of 10.

$$W \sim \mathrm{B}(10,\ 1-0.20608)$$
$$W \sim \mathrm{B}(10,\ 0.79392)$$

$$\mathrm{P}(W=10)=0.0995$$

Alternative method:

$$(1-0.20608)^{10}=0.0995$$

(v) Let V be the number of wasabi-flavoured sweets out of 200.

$$V \sim \mathrm{B}(200,\ 0.2)$$

$$\mathrm{P}(V\ge 30)=1-\mathrm{P}(V\le 29)=0.972$$

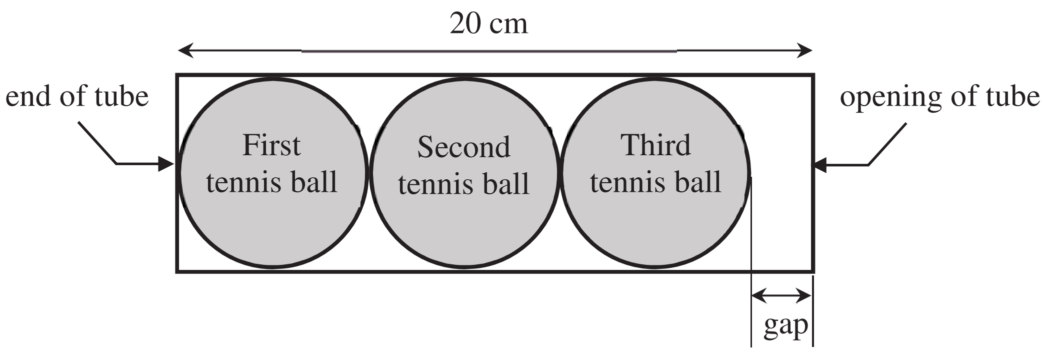
(vi) Part **(iv)** is a subset of part **(v)**, for example part (v) include cases where some packets have less than 3 wasabi sweets but overall the 10 packets have at least 30 wasabi sweets.

12 A company manufactures tennis balls and packs them into cylindrical tubes for sale. The tennis balls have radii that are normally distributed with mean 3.3 cm and standard deviation 0.2 cm.

(i) Find the probability that the radius of a randomly selected tennis ball lies between 3.135 cm and 3.465 cm. Without any further calculation, explain, with the aid of a diagram, how the answer obtained would compare with the probability that the radius lies between 3.465 cm and 3.795 cm. [3]

(ii) 3 tennis balls are randomly selected. Find the probability that exactly one of them has a radius less than 3.4 cm and two of them have radii greater than 3.4 cm each. [2]

The cylindrical tubes are 20 cm long. 3 tennis balls are randomly selected and packed into a cylindrical tube such that the first tennis ball is in contact with the end of the tube and each subsequent ball is in contact with its neighbouring ball as shown in the diagram below. Assume that the centres of all the tennis balls are horizontally aligned.



(iii) Find the probability that a gap exists between the third tennis ball and the opening of the tube. [3]

(iv) Find the range of values of k such that the probability that the gap between the third tennis ball and the opening of the tube is more than k cm is at most 0.15. [3]

State an assumption used in your calculations. [1]

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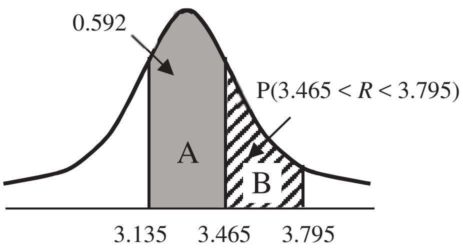
[Solution]

Let R be the radius of a randomly chosen tennis ball, $R \sim \mathrm{N}\left(3.3,\ 0.2^2\right)$

(i) $\mathrm{P}\left(3.135 < R < 3.465\right)$

$\approx 0.59063 = 0.591$ (3 sf)



From diagram, $\mathrm{P}\left(3.135 < R < 3.465\right) > \mathrm{P}\left(3.465 < R < 3.795\right)$

since area A is larger than area B given that the <u>widths</u> of the two intervals $3.135 < R < 3.465$ and $3.465 < R < 3.795$ are the <u>same</u>.

(ii) Required probability

$= \mathrm{P}\left(R < 3.4\right) \times \mathrm{P}\left(R > 3.4\right) \times \mathrm{P}\left(R > 3.4\right) \times 3$

$= (0.69146) \times (1 - 0.69146)^2 \times 3$

$= 0.197$ (3 sf)

(iii) Let G be the gap between the third tennis ball and the opening of the tube

$G = 20 - 2(R_1 + R_2 + R_3)$

$G \sim \mathrm{N}(\ 20 - 2(3.3 \times 3),\ 2^2(0.2^2 \times 3)\ )$

i.e. $G \sim \mathrm{N}(\ 0.2,\ \sqrt{0.48}^{\,2}\ )$

$\mathrm{P}(G > 0) = 0.614$ (3 s.f.)

<u>Alternative</u>

Let D be the diameters of 3 tennis balls

$D = 2(R_1 + R_2 + R_3)$

$D \sim \mathrm{N}(\ 2(3.3 \times 3),\ 2^2(0.2^2 \times 3)\ )$

i.e. $D \sim \mathrm{N}(\ 19.8,\ \sqrt{0.48}^{\,2}\ )$

$\mathrm{P}(D < 20) = 0.614$ (3 s.f.)

(iv) $\mathrm{P}(G > k) \le 0.15$

$1 - \mathrm{P}(G \le k) \le 0.15$

$\mathrm{P}(G \le k) \ge 0.85$

From GC,

$k \ge 0.918$ (3 sf)

Using invNorm function

Alternative

$\mathrm{P}(20 - D > k) \le 0.15$

$\mathrm{P}(D < 20 - k) \le 0.15$

From GC,

$20 - k \le 19.082$

$k \ge 0.918$ (3 sf)

Using invNorm function

The radii of the tennis balls are independent of one another.