Section A: Pure Mathematics [40 marks]

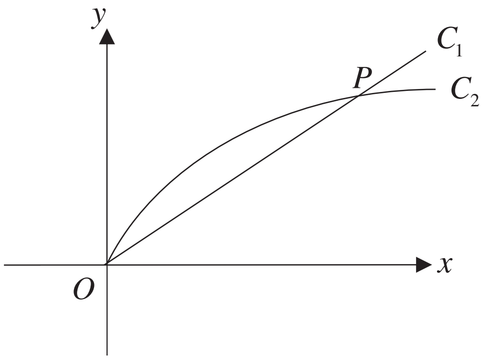
1. Find, algebraically, the range of values of k for which

$$kx^2 - 4x + k < 0$$

for all real values of x. [4]

2. Given that $2x^4 + x^2 - 1 = 0$, use the substitution $u = x^2$ to find the exact values of x. [4]

3.



The diagram shows the graphs of $C_1 : y = \dfrac{2x}{e-1}$ and $C_2 : y = \ln(2x+1)$ for $x \geq 0$. The graphs intersect at the origin O and the point P with coordinates $\left(\dfrac{e-1}{2}, k\right)$.

(i) Find the value of k. [1]

(ii) Find the exact area of the region bounded by C_2, the $y-$axis and the line $y = k$. [5]

(iii) Hence or otherwise, find the exact area of the region bounded by C_1 and C_2. [2]

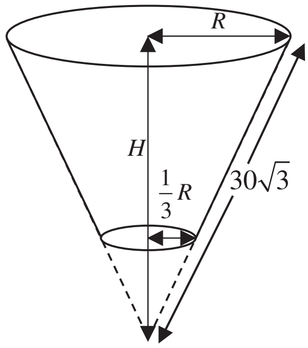
4. The curve C has equation $y = \dfrac{2}{3x+1} - e^{3x-1}$.

(i) Sketch the graph of C, stating the coordinates of any points of intersection with the axes and the equation of the asymptotes. [3]

(ii) Without using a calculator, find the equation of the tangent to C at the point where $x = \dfrac{1}{3}$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers to be determined. [4]

(iii) The tangent found in part **(ii)** meets the $y-$axis at A and $x-$axis at B. Find the length of AB. [4]

5. [It is given that a right circular cone of base radius r, perpendicular height h and slant height l has volume $\frac{1}{3}\pi r^2 h$ and curved surface area πrl.]



The diagram shows an open-top waste paper basket created by truncating an inverted right circular cone with negligible thickness. The larger inverted right circular cone has base radius R cm, perpendicular height H cm, and slant height $30\sqrt{3}$ cm. The bottom of the waste paper basket, where the smaller inverted right circular cone is truncated, has radius $\frac{1}{3}R$ cm.

(i) Show that the volume V cm^3, of the waste paper basket is given by

$$V = \frac{26}{81}\pi\left(2700H - H^3\right).$$ [3]

(ii) Without using a calculator, find the maximum value of V as H varies, justifying that this value is a maximum. [5]

An entrepreneur decides to manufacture the waste paper baskets using economy, standard and premium materials. Waste paper baskets manufactured from each material have different unit costs. The table below shows the number of waste paper baskets manufactured from each of the materials and the total costs based on records of 2015, 2016 and 2017.

	2015	**2016**	**2017**
Economy	200	300	350
Standard	170	200	300
Premium	50	60	70
Total cost in $	540.00	687.00	901.50

(iii) Assuming that the unit costs of the waste paper baskets for each material did not change from 2015 to 2018, find the total cost of manufacturing 100 economy, 70 standard and 50 premium waste paper baskets in 2018. [5]

Section B: Statistics [60 marks]

6. As part of Singapore's aim to be a Smart Nation in 10 years, hawker centres are encouraged to go cashless. An initial trial of cashless payment in hawker centres shows that 1 in 5 customers uses cashless payment. A random sample of 12 customers is selected and the number of customers who use cashless payment is denoted by the random variable C.

(i) Explain what is meant in this context by the term 'a random sample'. [1]

(ii) State, in context, an assumption needed for C to be well modelled by a binomial distribution. [1]

Assume now that C has a binomial distribution.

(iii) Find the probability that at least 3 customers use cashless payment. [2]

Eight groups of 12 customers are randomly selected.

(iv) Find the probability that there are exactly 5 groups with at least 3 customers who use cashless payment. [2]

7. A group of 11 people, consisting of 6 men and 5 women, stand in a line for photo-taking. There are 3 married couples in this group, with each married couple consisting of a husband and a wife. Find the number of different possible arrangements if

(i) there is no restriction, [1]

(ii) each married couple stand together, [2]

(iii) men and women alternate, [2]

(iv) a man stands on the extreme left or a woman stands on the extreme right or both. [3]

8. The number of international visitor arrivals in Singapore, x, in thousands, and the Gross Domestic Product (GDP) per capita, y, in thousands of dollars, were recorded for a sample of 8 countries. The results are given in the following table.

Country	Australia	Iran	Japan	Russian Federation	South Korea	Sri Lanka	Taiwan	United Kingdom
x	1,082	25	793	80	631	108	396	519
y	55.7	5.3	38.4	10.6	29.9	4.1	24.6	37.2

(i) Draw a sketch of the scatter diagram for the data, as shown on your calculator. [2]

(ii) Find the product moment correlation coefficient and comment on its value in the context of the data. [2]

(iii) Find the regression line of x on y and sketch this line on your scatter diagram. [3]

(iv) Use an appropriate regression line to calculate an estimate of the number of international visitor arrivals from Canada whose GDP per capita is 45100 dollars. Comment on the reliability of your estimate. [2]

9. Kickers chocolates are sold in tins of 5 chocolates. The masses, in grams, of the individual Kickers chocolates and the empty tins have independent normal distributions with means and standard deviations as shown in the following table.

	Mean	Standard Deviation
Individual Kickers Chocolate	53	2.8
Empty Tin	15	0.4

(i) Find the probability that two randomly chosen Kickers chocolate each weigh more than 50 grams. [1]

(ii) Find the probability that the total mass of a tin containing 5 Kickers chocolates is less than 275 grams. State the mean and variance of the distribution that you use. [3]

The masses, in grams, of the individual Venus chocolates have a normal distribution with mean 35 grams and standard deviation σ grams. It is given that 85% of Venus chocolates weigh more than 34 grams.

(iii) Find σ, giving your answer correct to 4 decimal places. [3]

The cost of producing Kickers and Venus chocolates is 2 cents per gram and 3 cents per gram respectively.

(iv) Find the probability that the cost of producing a Kickers chocolate is within 10 cents of the cost of producing a Venus chocolate. State an assumption needed for your calculation. [4]

10. In a game, Mr Lim and Mr Tan take turns to pick a ball, without replacement, from a box which contains 3 red balls, 2 green balls and 1 yellow ball. The game continues until the first player picks a red ball and wins the game. Mr Lim starts the game.

(i) Draw a tree diagram to represent the possible outcomes. [3]

(ii) Find the probability that Mr Tan wins the game. [2]

(iii) If Mr Lim wins the game, find the probability that he wins on his second turn. [3]

(iv) Determine if the events 'Mr Lim has 2 turns' and 'Mr Lim wins the game' are independent. [2]

Mr Lim and Mr Tan play the game 5 times.

(v) Find the probability that Mr Tan wins the game at most two times. [2]

11. A government introduces the carbon tax to encourage companies to reduce carbon dioxide emission and lessen the effect of global warming. A random sample of 60 companies is taken and the amount of annual carbon dioxide emission (in tonnes) are summarised by

$$\sum(x-20000)=286\ 800\ ,\ \sum(x-20000)^2=1\ 429\ 904\ 000\,.$$

(i) Find unbiased estimates of the population mean and variance. [3]

(ii) Test at the 5% significance level, whether there is evidence that the population mean amount of annual carbon dioxide emission is less than 25000 tonnes. [4]

(iii) What do you understand by the terms 'unbiased estimate' and '5% significance level' in the context of this question? [2]

A researcher takes another sample of 60 companies and the mean amount of annual carbon dioxide emission recorded is k tonnes. It is given that the population standard deviation is 2000 tonnes. A test at the 1% significance level indicates that the population mean amount of annual carbon dioxide emission does not differ from 25000 tonnes.

(iv) Find the set of values of k, giving your answer correct to the nearest integer. [5]

End of Paper

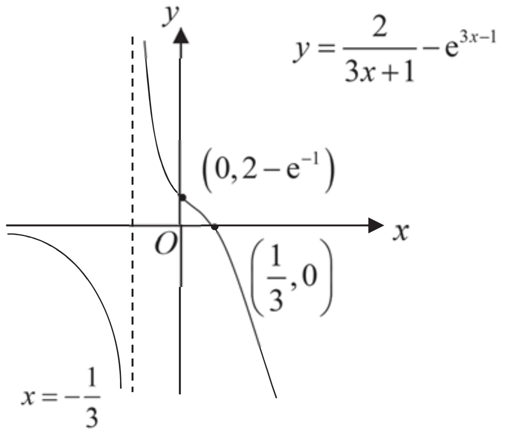
1	
	Since $kx^2-4x+k<0$, graph is "n-shape" and there are no real roots. Hence, $b^2-4ac<0$ and $k<0$ $(-4)^2-4(k)(k)<0$ $16-4k^2<0$ $4-k^2<0$ $(2+k)(2-k)<0$ $k<-2$ or $k>2$ $(\text{reject } \because k<0)$ Therefore, $k<-2$.

2	
	Substituting $u = x^2$, $2u^2 + u - 1 = 0$ $(u+1)(2u-1) = 0$ $u = -1$ or $u = \frac{1}{2}$ $\Rightarrow x^2 = -1 \ (\text{reject} \because x^2 \geq 0)$ or $x^2 = \frac{1}{2}$ $x^2 = \frac{1}{2}$ $x = \pm\frac{1}{\sqrt{2}}$

3		
	(i)	When $x=\dfrac{e-1}{2}$, $k=\dfrac{2\left(\dfrac{e-1}{2}\right)}{e-1}=1$
	(ii)	$y=\ln(2x+1)$ $e^{y}=2x+1$ $x=\dfrac{e^{y}-1}{2}$ $\int_0^1 \dfrac{e^{y}-1}{2}\,dy$ $=\dfrac{1}{2}\int_0^1 e^{y}-1\,dy$ $=\dfrac{1}{2}\left[e^{y}-y\right]_0^1$ $=\dfrac{1}{2}(e-1-1)$ $=\dfrac{1}{2}(e-2)$
	(iii)	Area $=\dfrac{1}{2}(1)\left(\dfrac{e-1}{2}\right)-\dfrac{1}{2}(e-2)$ $=\dfrac{1}{4}(3-e)$

4

(i)



(ii)

$\frac{dy}{dx}=-2(3)(3x+1)^{-2}-3e^{3x-1}$

When $x=\frac{1}{3}$, $\frac{dy}{dx}=-6(2)^{-2}-3e^{\frac{3}{3}-1}=-\frac{9}{2}$ and $y=1-e^{\frac{3}{3}-1}=0$

Equation of tangent:

$$y-0=-\frac{9}{2}\left(x-\frac{1}{3}\right)$$

$\frac{9}{2}x+y-\frac{3}{2}=0$

$9x+2y-3=0$

(iii)

At $y-$axis, $x=0$, $y=\frac{3}{2}$.

$\therefore A\left(0,\frac{3}{2}\right)$

At $x-$axis, $y=0$, $x=\frac{1}{3}$.

$\therefore B\left(\frac{1}{3},0\right)$

		Hence, $AB=\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{3}{2}\right)^2}$ $=\sqrt{\frac{85}{36}}$ $=1.5366$ $\approx 1.54 \quad (\text{to 3 s.f.})$

5

(i) By Pythagoras' theorem,

$$\left(30\sqrt{3}\right)^2 = R^2 + H^2$$

$$R^2 = 2700 - H^2$$

By similar triangles, truncated part of the cone has dimensions $\frac{1}{3}R$ and $\frac{1}{3}H$.

$$\begin{aligned} V &= \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi\left(\frac{1}{3}R\right)^2\left(\frac{1}{3}H\right) \\ &= \frac{1}{3}\pi R^2 H - \frac{1}{81}\pi R^2 H \\ &= \frac{26}{81}\pi R^2 H \\ &= \frac{26}{81}\pi\left(2700 - H^2\right)H \\ &= \frac{26}{81}\pi\left(2700H - H^3\right) \end{aligned}$$

(ii) At maximum, $\frac{\mathrm{d}V}{\mathrm{d}H} = 0$

$$\frac{26}{81}\pi\left(2700 - 3H^2\right) = 0$$

$$2700 - 3H^2 = 0$$

$$H^2 = 900$$

$$H = 30 \quad \because H > 0$$

H	30^-	30	30^+
$\frac{\mathrm{d}V}{\mathrm{d}H}$	>0	0	<0
	/	—	\

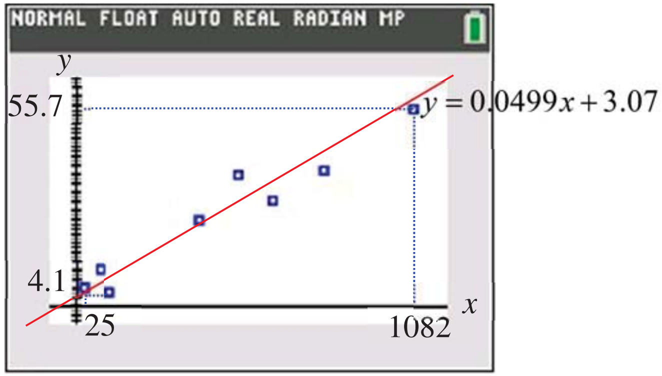
<u>Alternative:</u>

		$\dfrac{d^2V}{dh^2}=\dfrac{26}{81}\pi(-6H)$ <0 when $H=30$ Therefore, V is a maximum when $H=30$. $V=\dfrac{26}{81}\pi\left(2700(30)-(30)^3\right)$ $=\dfrac{52000\pi}{3}$
	(iii)	Let x, y and z represent the unit costs of manufacturing waster paper baskets of economy, standard and premium materials in dollars. $200x+170y+50z=540$ $300x+200y+60z=687$ $350x+300y+70z=901.5$ Using G.C. $x=0.75$ $y=1.5$ $z=2.7$ Cost in 2018: $100(\$0.75)+70(\$1.50)+50(\$2.70)=\$315$

6		
	(i)	A random sample is a sample where all customers have equal chance of being selected and the selection is done independently.
	(ii)	Whether or not a customer uses cashless payment is independent of other customers using cashless payment. OR The probability of customers using cashless payment remains constant at 0.2.
	(iii)	Let C be the random variable denoting the number of customers who use cashless payment, out of 12 customers. $\therefore C \sim \mathrm{B}(12,\ 0.2)$ $\mathrm{P}(C \geq 3) = 1 - \mathrm{P}(C \leq 2)$ $= 0.4416542515$ $\approx 0.442 \quad (\text{to 3 s.f.})$
	(iv)	Let Y be the random variable denoting the number of groups with at least 3 customers who uses cashless payment, out of 8 groups. $\therefore Y \sim \mathrm{B}(8,\ \mathrm{P}(C \geq 3))$ $\Rightarrow Y \sim \mathrm{B}(8,\ 0.4416542515)$ $\mathrm{P}(Y = 5) = 0.1637984338$ $\approx 0.164 \quad (\text{to 3 s.f.})$

7		
	(i)	No. of arrangements $= 11! = 39916800$
	(ii)	No. of arrangements $= 8! \times (2!)^3$ $= 322560$
	(iii)	No. of arrangements $= 6! \times 5!$ $= 86400$
	(iv)	<u>*Man on extreme left*</u> No. of arrangements $= 6 \times 10! = 21772800$ <u>*Woman on extreme right*</u> No. of arrangements $= 5 \times 10! = 18144000$ <u>*Man on extreme left AND woman on extreme right*</u> No. of arrangements $= 6 \times 5 \times 9! = 10886400$ Total no. of arrangements $= 21772800 + 18144000 - 10886400$ $= 29030400$ <u>**Alternative:**</u> *Man on extreme left and woman NOT on extreme right* No. of arrangements $= 6 \times 5 \times 9! = 10886400$ *Woman on extreme right and man NOT on extreme left* No. of arrangements $= 5 \times 4 \times 9! = 7257600$ *Man on extreme left AND woman on extreme right* No. of arrangements $= 6 \times 5 \times 9! = 10886400$

		Total no. of arrangements $= 10886400 + 7257600 + 10886400$ $= 29030400$

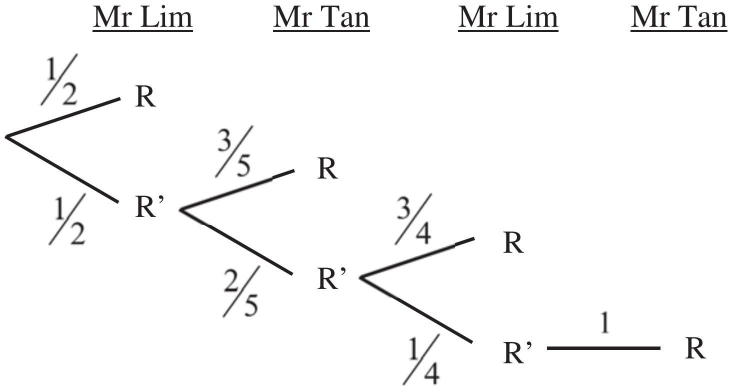
8		
	(i)	
	(ii)	Using GC, $r = 0.970220531 \approx 0.970\ (3\text{ s.f.})$. As the value of r is close to 1, this suggests that there is a strong, positive, linear relationship between the number of international visitor arrivals and GDP per capita. As the number of international visitor arrivals increases, GDP per capita increases proportionally.
	(iii)	Using GC, regression line of x on y is $x = 20.04745783y - 61.47085271 \approx 20.0 - 61.5 \quad (\text{to 3 s.f.})$ Making y the subject, $y = \dfrac{x + 61.47085271}{20.04745783} = 0.0498816363x + 3.066266717$ $\approx 0.0499x + 3.07\,(3\text{ s.f.})$
	(iv)	$y = \dfrac{45100}{1000} = 45.1$ When $y = 45.1$, $x = 20.04745783(45.1) - 61.47085271$ $= 842.6694954$ $= 843\,(\text{to 3 s.f.})$
		The estimated number of visitor arrivals from Canada is 843000. This estimate is reliable as $r = 0.970$ is close to 1 and $y = 45.1$ is within the data range of $4.1 \le y \le 55.7$.

9		
	(i)	Let K be the random variable denoting the mass, in grams, of a randomly selected Kickers chocolate. $\therefore K \sim \mathrm{N}\left(53,\ 2.8^2\right)$ $\mathrm{P}\left(K_1 > 50\right) \times \mathrm{P}\left(K_2 > 50\right) = 0.7361838758$ $\approx 0.736 \quad (\text{to 3 s.f.})$
	(ii)	Let T be the random variable denoting the mass, in grams, of a randomly selected empty tin. $\therefore T \sim \mathrm{N}\left(15,\ 0.4^2\right)$ $K_1 + K_2 + K_3 + K_4 + K_5 + T \sim \mathrm{N}\left(5\times53+15,\ 5\times2.8^2+0.4^2\right)$ $K_1 + K_2 + K_3 + K_4 + K_5 + T \sim \mathrm{N}\left(280,\ 39.36\right)$ $\mathrm{P}\left(K_1 + K_2 + K_3 + K_4 + K_5 + T < 275\right) = 0.2127339029$ $\approx 0.213 \quad (\text{to 3 s.f.})$
	(iii)	Let V be the random variable denoting the mass, in grams, of a randomly selected Venus chocolate. $\therefore V \sim \mathrm{N}\left(35,\ \sigma^2\right)$ $\mathrm{P}\left(V > 34\right) = 0.85$ $\mathrm{P}\left(V < 34\right) = 0.15$ $\mathrm{P}\left(Z < \frac{34-35}{\sigma}\right) = 0.15$ $\mathrm{P}\left(Z < \frac{-1}{\sigma}\right) = 0.15$ $\frac{-1}{\sigma} = -1.03643338$ $\sigma = 0.9648473501 \approx 0.9648 \quad (\text{to 4 d.p.})$
	(iv)	$2K \sim \mathrm{N}\left(2\times53,\ 2^2\times2.8^2\right)$ $2K \sim \mathrm{N}\left(106,\ 31.36\right)$ $3V \sim \mathrm{N}\left(3\times35,\ 3^2\times0.9648473501^2\right)$ $3V \sim \mathrm{N}\left(105,\ 8.378373681\right)$

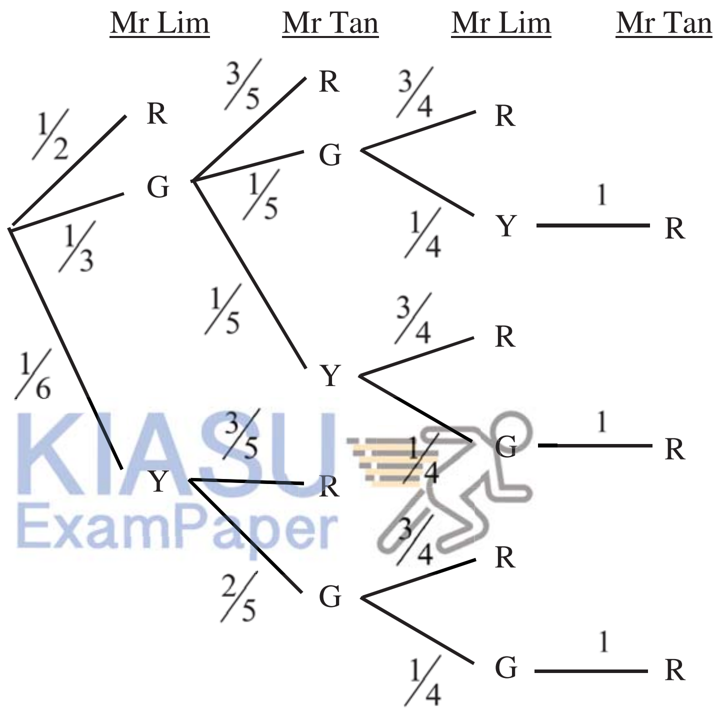
		$2K - 3V \sim \mathrm{N}(106 - 105,\ 31.36 + 8.378373681)$ $2K - 3V \sim \mathrm{N}(1,\ 39.73837368)$ $\mathrm{P}(-10 < 2K - 3V < 10) = 0.8828159632 \approx 0.883 \quad (\text{to 3 s.f.})$
		The mass of the Kickers chocolates is independent of the mass of the Venus chocolates.

10

(i)



Alternative:



	(ii)	$P(\text{Mr Tan wins the game})$ $=\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{2}{5}\times\frac{1}{4}\times 1$ $=0.35$ **Alternative:** $P(\text{Mr Tan wins the game})$ $=\frac{1}{3}\times\frac{3}{5}+\frac{1}{6}\times\frac{3}{5}+\frac{1}{3}\times\frac{1}{5}\times\frac{1}{4}\times 1+\frac{1}{3}\times\frac{1}{5}\times\frac{1}{4}\times 1+\frac{1}{6}\times\frac{2}{5}\times\frac{1}{4}\times 1$ $=0.35$
	(iii)	$P(\text{Mr Lim wins game on his second turn} \mid \text{Mr Lim wins the game})$ $=\dfrac{P(\text{Mr Lim wins the game on his second turn})}{P(\text{Mr Lim wins the game})}$ $=\dfrac{\frac{1}{2}\times\frac{2}{5}\times\frac{3}{4}}{1-0.35}$ $=\frac{0.15}{0.65}=\frac{3}{13}$ **Alternative:** $P(\text{Mr Lim wins game on his second turn} \mid \text{Mr Lim wins the game})$ $=\dfrac{P(\text{Mr Lim wins the game on his second turn})}{P(\text{Mr Lim wins the game})}$ $=\dfrac{\frac{1}{3}\times\frac{1}{5}\times\frac{3}{4}+\frac{1}{3}\times\frac{1}{5}\times\frac{3}{4}+\frac{1}{6}\times\frac{2}{5}\times\frac{3}{4}}{1-0.35}$ $=\frac{0.15}{0.65}=\frac{3}{13}$
	(iv)	$P(\text{Mr Lim has two turns})=\frac{1}{2}\times\frac{2}{5}=0.2$

		$\mathrm{P}(\text{Mr Lim wins the game}) = 0.65$ $\mathrm{P}(\text{Mr Lim has two turns and wins the game}) = 0.15$ Since $0.2 \times 0.65 = 0.13 \neq 0.15$, the two events are not independent.
	(v)	Let X be the random variable denoting the number of games Mr Tan wins, out of 5 games. $\therefore X \sim \mathrm{B}(5,\ 0.35)$ $\mathrm{P}(X \leq 2) = 0.764830625 \approx 0.765 \quad (\text{to 3 s.f.})$

11

(i)

$$\hat{\mu} = \frac{286800}{60} + 20000 = 24780$$

$$s^2 = \frac{1}{59}\left[1429904000 - \frac{286800^2}{60}\right] = 1000000$$

(ii)

Let X be the random variable denoting the amount of annual carbon dioxide emission, in tonnes, of a randomly selected company.

$H_0 : \mu = 25000$ tonnes

$H_1 : \mu < 25000$ tonnes

Under H_0, $\mu = 25000$ tonnes, and since $n = 60$ is large enough, by Central Limit Theorem, $\bar{X} \sim N\left(25000, \frac{1000000}{60}\right)$ approximately.

Test statistic, $z = \dfrac{24780 - 25000}{\sqrt{\dfrac{1000000}{60}}} = -1.704112672$

$p\text{-value} = P(Z < -1.704112672)$

$= 0.0441799877$

Method 1:

At 5% significance level, reject H_0 if $p\text{-value} < 0.05$.

Since $p\text{-value} = 0.0441799877 < 0.05$, reject H_0.

Method 2:

At 5% significance level, reject H_0 if $z < -1.644853626$.

Since $z = -1.704112672 < -1.644853626$, reject H_0.

Conclude that there is sufficient evidence at the 5% significance level that the population mean amount of annual carbon dioxide emission is less than 25000 tonnes.

(iii)

An unbiased estimate is the value of an estimator for the population mean annual carbon dioxide emission where the expected value of the estimator is equal to the population mean annual carbon dioxide emission, i.e. $E(\theta) = \mu$ where θ is the estimator.

5% significance level means that there is a probability of 0.05 of claiming that the population mean amount of annual carbon dioxide emission is less than 25000 tonnes, when it is actually 25000 tonnes.

(iv)

$H_0 : \mu = 25000$ tonnes

$H_1 : \mu \neq 25000$ tonnes

Under H_0, $\mu = 25000$ tonnes, and since $n = 60$ is large enough, by Central Limit Theorem, $\bar{X} \sim N\left(25000, \dfrac{2000^2}{60}\right)$ approximately.

Test statistic, $z = \dfrac{k - 25000}{\dfrac{2000}{\sqrt{60}}}$

At 1% significance level, reject H_0 if $z < -2.575829303$ or $z > 2.575829303$.

Since H_0 is not rejected,

$$-2.575829303 < \frac{k-25000}{\frac{2000}{\sqrt{60}}} < 2.575829303$$

$$-2.575829303\left(\frac{2000}{\sqrt{60}}\right) < k - 25000 < 2.575829303\left(\frac{2000}{\sqrt{60}}\right)$$

$$25000 - 2.575829303\left(\frac{2000}{\sqrt{60}}\right) < k < 25000 + 2.575829303\left(\frac{2000}{\sqrt{60}}\right)$$

$$24334.92373 < k < 25665.07627$$

$$\Rightarrow \{k \in \mathbb{Z} : 24335 \le k \le 25665\}$$