1. Equations and inequalities

Rational inequalities

Approach

- Step 1: Make one side of the inequality 0 by addition/subtraction. **Do not cross multiply!**
- Step 2: Combine into a single fraction and factorize as much as possible.
- Step 3: If a quadratic has no real roots, complete the square to conclude it is always positive/negative.
- Step 4: Use a number line to obtain the solution.

Example

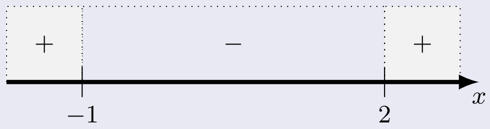
Solve $\dfrac{3x+5}{x^2-x-2} > -1$.

Step 1:
$$\frac{3x+5}{x^2-x-2} > -1$$
$$\frac{3x+5}{x^2-x-2} + 1 > 0$$

Step 2,3:
$$\frac{x^2+2x+3}{x^2-x-2} > 0$$
$$\frac{(x+1)^2+2}{(x+1)(x-2)} > 0$$

Step 3:
Since $(x+1)^2+2$
is always positive for all $x \in \mathbb{R}$,
$$\frac{1}{(x+1)(x-2)} > 0$$

Step 4:



Solution: $x < -1$ or $x > 2$

Modulus inequalities

Formula

Let k be a positive real number.

$$\boxed{|x| < k} \quad \Rightarrow \quad \boxed{-k < x < k}$$
$$\boxed{|x| > k} \quad \Rightarrow \quad \boxed{x < -k \quad \text{or} \quad x > k}$$

Graphing calculator techniques

Approach

- We can use the "PlySmlt2" app to solve **systems of linear equations**.
- The "Plysmlt2" app can also be used to solve polynomials like $2x^3 - x^2 + 2x - 1 = 0$.
- We can use the graph of a curve and the "zero" solver to solve equations like $\ln x - x^2 = 0$.
- We can use the graph of two curves and the "intersect" solver to solve equations like $\ln x = x^2$.
- For non-rational inequalities like $e^x > 3x$, we can use a GC to sketch two curves and locate the region(s) where one curve is higher than the other.
  Alternatively, we can sketch one curve by rearranging the inequality to $e^x - 3x > 0$.

The discriminant

Approach

- Step 1: Cross multiply and form a quadratic equation in terms of x.
- Step 2: For the set of values that y can take, the **discriminant** $b^2 - 4ac \geq 0$.

Example

$$y = \frac{x+1}{x^2+x+1}.$$

Use an algebraic method to find the set of values that y can take.

Step 1:
$y = \dfrac{x+1}{x^2+x+1}$
$(x^2+x+1)y = x+1$
$yx^2 + yx + y - x - 1 = 0$
$yx^2 + (y-1)x + (y-1) = 0$

Step 2:
For set of values that y can take,
$(y-1)^2 - 4(y)(y-1) \geq 0$
$-3y^2 + 2y + 1 \geq 0$
$(3y+1)(-y+1) \geq 0$
Solution: $-\frac{1}{3} \leq y \leq 1$

2. Functions

Basics

Definition

- The **domain**, D_f, of a function, f, refers to the set of all possible "inputs" (typically "x").
- The **range**, R_f, of a function, f, refers to the set of all possible "outputs" (typically "y").
- Drawing graphs is very useful in finding the range of functions.

Set and interval notation

Example

- We can refer to the set of all real numbers using $\mathbb{R}$, $\{x : x \in \mathbb{R}\}$ or $(-\infty, \infty)$.
- We can refer to all the real numbers from -5 (inclusive) to 3 (non-inclusive) by $\{x \in \mathbb{R} : -5 \leq x < 3\}$ or $[-5, 3)$.
- We can refer to all the real numbers except 1 by $\mathbb{R} \setminus \{1\}$ or $\{x \in \mathbb{R} : x \neq 1\}$ or $(-\infty, 1) \cup (1, \infty)$.

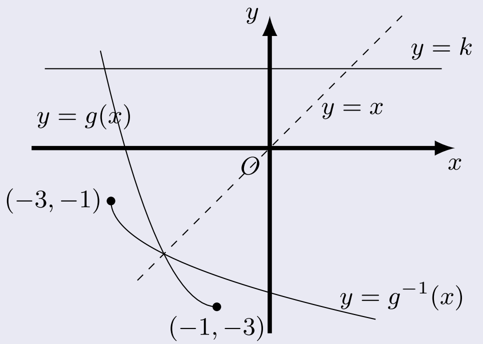
Inverse functions

Theory

- A function has an **inverse**, denoted by f^{-1}, if f is one-one.
- We can determine if f is one-one by employing the **horizontal line test**.
- To find the rule for f^{-1}, we let $y = f(x)$ and make x the subject.
- $\boxed{D_{f^{-1}} = R_f}$, $\boxed{R_{f^{-1}} = D_f}$.
- The graph of $y = f^{-1}(x)$ can be obtained by reflecting the graph of $y = f(x)$ in the line $y = x$.

Example

$$f : x \mapsto x^2 + 2x - 2, \qquad x \in \mathbb{R}$$
$$g : x \mapsto x^2 + 2x - 2, \qquad x \in \mathbb{R}, x \leq -1$$



The horizontal line $y = 1$ cuts the curve of $y = f(x)$ more than once. Hence f is not one-one and f^{-1} does not exist.

All horizontal lines $y = k, k \in \mathbb{R}$ cut the curve of $y = g(x)$ **at most once**. Hence g is one-one and g^{-1} exists.

$y = x^2 + 2x - 2$
$y = (x + 1)^2 - 3$
$(x + 1)^2 = y + 3$
$x + 1 = \pm\sqrt{y + 3}$
Since $x \leq -1$,
$x = -1 - \sqrt{y + 3}$.

$D_{g^{-1}} = R_g = [-3, \infty)$.
$g^{-1} : x \mapsto -1 - \sqrt{x + 3}, \quad x \in \mathbb{R},\ x \geq -3$.

Composite functions

Theory

- The **composite function** fg consists of first applying g followed by f.
- fg exists if $\boxed{R_g \subseteq D_f}$. fg does not exist if $\boxed{R_g \not\subseteq D_f}$.
- $\boxed{D_{fg} = D_g}$.
- To find R_{fg}, we draw the graphs of $y = f(x)$ and $y = g(x)$ separately. Find R_g first, and then use R_g as the domain of f to obtain R_{fg}.

Example

$$f : x \mapsto \ln x, \qquad x \in \mathbb{R},\ 0 < x < 2$$
$$g : x \mapsto x^2 - 1, \qquad x \in \mathbb{R}$$

$R_g = [-1, \infty),\ D_f = (0, 2)$.
$R_f = (-\infty, \ln 2),\ D_g = (-\infty, \infty)$.

$R_g \not\subseteq D_f \Rightarrow fg$ does not exist
$R_f \subseteq D_g \Rightarrow gf$ exists

$gf(x) = g(\ln x) = (\ln x)^2 - 1$

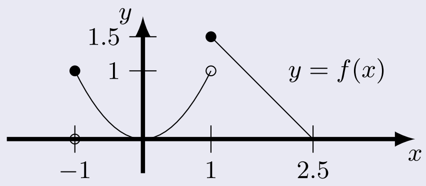
$D_{gf} = D_f = (0, \infty)$
$R_{gf} = [-1, \infty)$

Special examples

$ff^{-1}(x) = x, \quad f^{-1}f(x) = x, \quad f^2(x) = ff(x), \quad$ **periodic** function: $f(x + a) = f(x)$.

A piecewise function

$$f(x) = \begin{cases} x^2 & \text{for } x - 1 \leq x < 1 \\ 2.5 - x & \text{for } 1 \leq x \leq 2.5 \\ 0 & \text{otherwise} \end{cases}$$



3. Graphs and transformations

Asymptotes

Theory

- $y = a + \dfrac{b}{cx+d}$ has asymptotes $\boxed{x = -\dfrac{d}{c}}$ and $\boxed{y = a}$.
- $y = Ax + B + \dfrac{E}{Cx + D}$

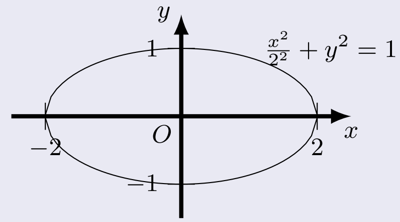
  has a vertical asymptote $\boxed{x = -\dfrac{D}{C}}$ and an **oblique** asymptote $\boxed{y = Ax + B}$.
- **Long division** is a useful technique to obtain the forms above.

Conics

Ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

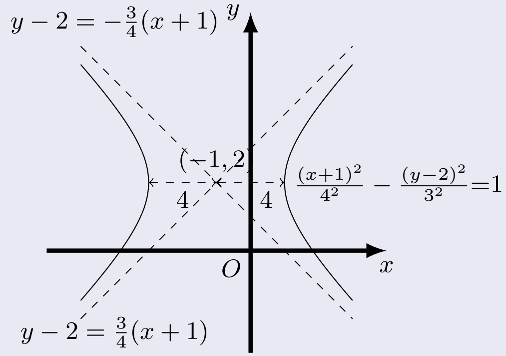
Center (h, k),
horizontal radius a,
vertical radius b.



Hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

Center (h, k).



Completing the square is a useful technique to obtain the forms above.

Modulus transformations

Theory

- To sketch $\boxed{y = |f(x)|}$, reflect the parts of $y = f(x)$ that are below the x-axis.
- To sketch $\boxed{y = f(|x|)}$,
  - Step 1: Remove the parts of $y = f(x)$ to the left of the y-axis.
  - Step 2: Reflect the parts of the graph to the right about the y-axis.

Basic transformations (translation, scaling, reflection)

Transformation	Equation
translate a units in positive x-axis direction	replace x with $x - a$ $f(x) \to f(x-a)$
scale with scale factor b parallel to the x-axis	replace x with $\frac{x}{b}$ $f(x) \to f(\frac{x}{b})$
reflect in y-axis	replace x with $-x$ $f(x) \to f(-x)$
translate A units in positive y-axis direction	replace y with $y - A$ $f(x) \to f(x) + A$
scale with scale factor B parallel to y-axis	replace y with $\frac{y}{B}$ $f(x) \to Bf(x)$
reflect in x-axis	replace y with $-y$ $f(x) \to -f(x)$

Order matters. For example,
$f(x) \to f(x+1) \to f(2x+1)$ vs $f(x) \to f(2x) \to f(2(x+1))$.

Further transformations $\left(y = \frac{1}{f(x)}, y = f'(x)\right)$

Approach

$y = f(x)$	$y = \frac{1}{f(x)}$	$y = f'(x)$
horizontal asymptote $y = k$	horizontal asymptote $y = \frac{1}{k}$	horizontal asymptote $y = 0$
oblique asymptote $y = mx + c$	horizontal asymptote $y = 0$	horizontal asymptote $y = m$
vertical asymptote $x = a$	x-intercept $(a, 0)$	vertical asymptote $x = a$
x-intercept $(b, 0)$	vertical asymptote $x = b$	-
y-intercept $(0, d)$	y-intercept $(0, \frac{1}{d})$	-
max/min point (A, B)	min/max point $(A, \frac{1}{B})$	x-intercept $(A, 0)$
y increasing/decreasing	y decreasing/increasing	y positive/negative
y positive/negative	y positive/negative	-
slope increasing in magnitude	-	y increasing in magnitude

4. Arithmetic and geometric progressions. 5. The sigma notation

Basics

Theory

- Let u_n denote the nth term of a **sequence**.
- Let S_n denote the sum of the first n terms of a **series**. $S_n = u_1 + u_2 + \ldots + u_{n-1} + u_n$.
- A sequence **converges** if u_n gets arbitrarily close to a finite number when n gets very large. We write $u_n \to a$ as $n \to \infty$, or $\lim_{n\to\infty} u_n = a$. a is called the **limit** of the sequence.
- A series **converges** if S_n gets arbitrarily close to a finite number when n gets very large. We write $S_n \to b$ as $n \to \infty$, or $\lim_{n\to\infty} S_n = S_\infty = b$. b is called the **limit** of the series.
- If a sequence/series does not converge, it is said to **diverge**
- To recover u_n from S_n: $\boxed{u_n = S_n - S_{n-1}}$.

Arithmetic progressions (APs)

Formula

- Let a be the **first term** of an AP and d the **common difference**.
- To prove that a sequence/series is arithmetic, we prove that $\boxed{u_n - u_{n-1} = \text{constant}}$.
- $\boxed{u_n = a + (n-1)d}$.
- $\boxed{S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(a + u_n)}$.

Method of differences

The method of differences can be used for sums like $\sum_{r=1}^{n} \frac{1}{r(r+1)}$. Partial fractions is often useful.

Geometric progressions (GPs)

Formula

- Let a be the **first term** of an GP and r the **common ratio**.
- To prove that a sequence/series is geometric, we prove that $\boxed{\frac{u_n}{u_{n-1}} = \text{constant}}$.
- $\boxed{u_n = ar^{n-1}}$.
- $\boxed{S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n - 1)}{r-1}}$.
- If $\boxed{-1 < r < 1}$, then a GP **converges** and **sum to infinity** $\boxed{S_\infty = \frac{a}{1-r}}$.

Sigma notation

Theory

- Example: $\sum_{r=3}^{5} f(r) = f(3) + f(4) + f(5)$.
- There are $\boxed{b-a+1}$ terms in $\sum_{r=a}^{b} f(r)$.
- Sum of a **constant**, $\sum_{r=a}^{b} k = (b-a+1)k$.
- $\sum_{r=a}^{b}(cr+d)$ is an AP, $\sum_{r=a}^{b} c^r$ is a GP.
- $\sum_{r=a}^{b}(kf(r) \pm g(r)) = k\sum_{r=a}^{b} f(r) \pm \sum_{r=a}^{b} g(r)$.
- $\sum_{r=a}^{c} f(r) = \sum_{r=a}^{b} f(r) + \sum_{r=b+1}^{c} f(r)$ where $a \le b < c$.

Change of variable

Example

Suppose $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$. Find $\sum_{r=2}^{n}(r+1)^2$.

Step 1a: Replace r with $r-1$.
Step 1b: Lower limit: $r - 1 = 2 \Rightarrow r = 3$.
Step 1c: Since lower limit increases by 1, upper limit also increases by 1.
Step 2: Split up the summation so that the lower limit matches what we know.
Step 3: Apply known formula.

$$\sum_{r=2}^{n}(r+1)^2 = \sum_{r=3}^{n+1} r^2 = \sum_{r=1}^{n+1} r^2 - \sum_{r=1}^{2} r^2 = \frac{(n+1)(n+2)(2n+3)}{6} - \frac{(2)(3)(5)}{6}$$

Savings and interest

Example

If we deposit \$2 in a bank at the start of every year and the bank gives 3% compound interest per annum at the end of every year, how much will we have in the bank at the end of n years?

Amount in bank:

Year	Start	End
1	2	$1.03(2)$
2	$2 + 1.03(2)$	$1.03(2) + 1.03^2(2)$
3	$2 + 1.03(2) + 1.03^2(2)$	$1.03(2) + 1.03^2(2) + 1.03^3(2)$
...	...	...
n	...	$1.03(2) + 1.03^2(2) + \ldots + 1.03^n(2)$

The amount in the bank forms a geometric series with first term $1.03(2)$ and common ratio 1.03.

$S_n = \frac{1.03(2)(1.03^n - 1}{1.03-1}$
Amount at end of n years: $\frac{206}{3}(1.03^n - 1)$.

6. Differentiation and applications

Basics

Theory

- A GC can perform **numerical differentiation**.
- A curve is (strictly) **increasing** if $f'(x) > 0$.
  A curve is (strictly) **decreasing** if $f'(x) < 0$.
- A curve is **concave up** if $f''(x) > 0$.
  A curve is **concave down** if $f''(x) < 0$.

Tangents/normals

Formula

- Equation of tangent/normal:
  $\boxed{y - y_1 = m(x - x_1)}$
  where m is the gradient of the tangent/normal at the point (x_1, y_1).
- Tangents vs normals: $\boxed{m_1 m_2 = -1}$.

Implicit differentiation

Example

Differentiate $(y^2 + 5x)^5 + x^2 y - \ln y = 8$ implicitly.

$$5(y^2 + 5x)^4 (2y\frac{\mathrm{d}y}{\mathrm{d}x} + 5) + 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Parametric equations

Formula

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}}$$

Curve sketching

- A GC can sketch parametric curves.
- Remember to change the "tMin" and "tMax" in the "window" screen!

Example

$x = t^2, \quad y = \sin t.$

Find the equation of the normal when $t = \pi$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos t}{2t}$
At $t = \pi, x = \pi^2$, $y = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2\pi}$
Equation of normal: $y = 2\pi(x - \pi^2)$.

Example

The point $\left(\frac{1}{t}, 2t^2 - 3\right)$ forms a curve as t varies. Find the cartesian equation of the curve.

$x = \frac{1}{t} \Rightarrow t = \frac{1}{x}$.
$y = 2t^2 - 3$.

Substituting $t = \frac{1}{x}$, the cartesian equation is $y = \frac{2}{x^2} - 3$

Maxima/minima, stationary points

Approach

- $\boxed{f'(x) = 0}$ at stationary points.
- For problem sums:
  - Step 1: Let the quantity to be maximized/minimized be A, for example. Find a formula for A involving other variables.
  - Step 2: If necessary, form other equations and manipulate so that the formula for A is in terms of only one variable (x for example).
  - Step 3: Differentiate to get $\frac{\mathrm{d}A}{\mathrm{d}x}$.
  - Step 4: At stationary values, $\boxed{\frac{\mathrm{d}A}{\mathrm{d}x} = 0}$. Solve for x.
  - Step 5: Answer the question by finding the required quantities.
  - Step 6: Prove that A is maximum/minimum.

Example: first derivative test

x	b^-	b	b^+
$f'(x)$	< 0	$= 0$	> 0
Shape	＼	—	／

Minimum at $x = b$.

Second derivative test

$f''(b)$	Conclusion
$f''(b) > 0$	minimum
$f''(b) < 0$	maximum
$f''(b) = 0$	no conclusion

Rates of change

Approach

- Rate of change of A: $\frac{\mathrm{d}A}{\mathrm{d}t}$.
- For problem sums:
  - Step 1: Translate the rates given in the problem sum.
  - Step 2: Form an equation between two related variables (A and x, for example).
  - Step 3: Differentiate to get $\frac{\mathrm{d}A}{\mathrm{d}x}$.
  - Step 4: Apply the chain rule expression $\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}}$.
  - Step 5: Answer the question by finding the required quantities.

7. Maclaurin series

In MF26

Formula

1 $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots$

2 $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$ $(|x| < 1)$

3 $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$ (all x)

4 $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots$ (all x)

5 $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots$ (all x)

6 $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3!} - \ldots + \frac{(-1)^{r+1}x^r}{r} + \ldots$ $(-1 < x \le 1)$

Maclaurin series using differentiation

Approach

- To get the Maclaurin series up until and including the term in x^n, differentiate n times.
- **Implicit differentiation** is often very useful.
- Sub in $x = 0$ to obtain $y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}$.
- Use formula 1 to obtain the Maclaurin series.

Approximations using Maclaurin series

When x is "small", we can often omit large powers of x in the Maclaurin series and still arrive at a reasonably accurate approximation.

Standard series

- Formulas 2-6 are often referred to as **standard series**.
- Start from "inside" and work outwards.

Example

- Expand $\ln(\cos x)$ up to and including the term in x^4.

$$\begin{aligned}\ln(\cos x) &\approx \ln\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)\\ &\approx \left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right) - \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right)^2\\ &\approx -\frac{x^2}{2} + \frac{x^4}{24} - \frac{1}{2}\left(\frac{x^4}{4}\right)\\ &\approx -\frac{x^2}{2} - \frac{x^4}{12}\end{aligned}$$

Small angle approximations

Formula

- $\sin x \approx x$
- $\cos x \approx 1 - \frac{x^2}{2}$
- $\tan x \approx x$

The **small angle approximations** for sine and cosine can be obtained by using formulas 4 and 5 up to and including the term in x^2.

Binomial expansion

Theory

- Formula 6 is often referred to as the **binomial expansion**.
- $|x| < 1$ is the **range of validity** of the expansion: if x is within the range of validity, then the Maclaurin series **converge** to $(1+x)^n$ as $r \to \infty$.

Example

Find the first three terms in the series expansion of $\frac{1}{\sqrt{2+x}}$.
What is the range of validity?

Range of validity: $\left|\frac{x}{2}\right| < 1 \Rightarrow -2 < x < 2$.

$$\begin{aligned}\frac{1}{\sqrt{2+x}} &= (2+x)^{-\frac{1}{2}}\\ &= 2^{-\frac{1}{2}}\left(1 + \frac{x}{2}\right)^{-\frac{1}{2}}\\ &= \frac{1}{\sqrt{2}}\left(1 - \frac{1}{2}\left(\frac{x}{2}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(\frac{x}{2}\right)^2 + \ldots\right)\\ &= \frac{1}{\sqrt{2}}\left(1 - \frac{x}{4} + \frac{3x^4}{32} + \ldots\right)\end{aligned}$$

8. Integration techniques

$f'(x)$ formulas

Formula

$f(x)$	$\int f(x)\,\mathrm{d}x$
$\dfrac{f'(x)}{f(x)}$	$\ln\lvert f(x)\rvert$
$f'(x)\big(f(x)\big)^n$	$\dfrac{1}{n+1}\big(f(x)\big)^{n+1}$
$f'(x)\mathrm{e}^{f(x)}$	$\mathrm{e}^{f(x)}$

$n \neq -1$ for the second formula.

Formulas in MF26

Formula

$f(x)$	$\int f(x)\,\mathrm{d}x$
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$

Algebraic techniques

Theory

- Long division.
- Partial fraction.
- "Forcing" terms.
- Completing the square.

Example

$$\begin{aligned}\int \frac{x^2+4x+8}{x^2+2x+5}\,\mathrm{d}x &= \int 1+\frac{2x+3}{x^2+2x+5}\,\mathrm{d}x\\ &= \int 1+\frac{2x+2}{x^2+2x+5}+\frac{1}{x^2+2x+5}\,\mathrm{d}x\\ &= \int 1+\frac{2x+2}{x^2+2x+5}+\frac{1}{(x+1)^2+2^2}\,\mathrm{d}x\\ &= x+\ln(x^2+2x+5)+\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C.\end{aligned}$$

Other formulas in MF26

$\int \tan x\,\mathrm{d}x = \ln\lvert\sec x\rvert$, $\int \sec x\,\mathrm{d}x = \ln\lvert\sec x+\tan x\rvert$. Formulas for $\cot x$ and $\operatorname{cosec} x$ are also provided.

Trigonometric techniques

Formula

- $\sin 2A = 2\sin A\cos A$.
- $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$.
- $\tan^2 A + 1 = \sec^2 A$.
- $\sin A\cos B = \frac{1}{2}\big(\sin(A+B)+\sin(A-B)\big)$.
- $\cos A\sin B = \frac{1}{2}\big(\sin(A+B)-\sin(A-B)\big)$.
- $\sin A\sin B = \frac{1}{2}\big(\cos(A+B)+\cos(A-B)\big)$.
- $\cos A\cos B = -\frac{1}{2}\big(\cos(A+B)+\cos(A-B)\big)$.

$$\int \sin^2 x\,\mathrm{d}x = \int \frac{1+\cos 2x}{2}\,\mathrm{d}x$$

$$\begin{aligned}\int \sqrt{1+\cos x}\,\mathrm{d}x &= \int \sqrt{1+2\cos^2\frac{x}{2}-1}\,\mathrm{d}x\\ &= \int \sqrt{2}\cos\frac{x}{2}\,\mathrm{d}x\end{aligned}$$

$$\int \tan^2 x\,\mathrm{d}x = \int \sec^2 x - 1\,\mathrm{d}x$$

$$\int \sin 5x\cos 3x\,\mathrm{d}x = \int \frac{1}{2}(\sin 8x+\sin 2x)\,\mathrm{d}x$$

Integration by parts

$$\boxed{\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int \frac{\mathrm{d}u}{\mathrm{d}x}v\,\mathrm{d}x}$$

$$\int x\cos x\,\mathrm{d}x = x\sin x - \int 1\cdot\cos x\,\mathrm{d}x.$$

"LIATE" heuristic

- **LI**: We typically differentiate (i.e. let them be "u") **logarithms** and **inverse trigonometric** functions.
- **TE**: We typically integrate (i.e. let them be "$\frac{\mathrm{d}v}{\mathrm{d}x}$") **trigonometric** functions and **exponential** functions.
- **A**: **Algebraic** terms typically depend on who they are paired with.

Example: integration by parts

$$\begin{aligned}\int (\sin x)\mathrm{e}^x\,\mathrm{d}x &= (\sin x)\mathrm{e}^x - \int (\cos x)\mathrm{e}^x\,\mathrm{d}x\\ &= (\sin x)\mathrm{e}^x - \left((\cos x)\mathrm{e}^x - \int (-\sin x)\mathrm{e}^x\,\mathrm{d}x\right)\\ &= (\sin x)\mathrm{e}^x - (\cos x)\mathrm{e}^x - \int (\sin x)\mathrm{e}^x\,\mathrm{d}x\\ 2\int (\sin x)\mathrm{e}^x\,\mathrm{d}x &= (\sin x)\mathrm{e}^x - (\cos x)\mathrm{e}^x + C\\ \int (\sin x)\mathrm{e}^x\,\mathrm{d}x &= \frac{1}{2}\Big((\sin x)\mathrm{e}^x - (\cos x)\mathrm{e}^x\Big) + C\end{aligned}$$

Integration by substitution

Approach

- Step 1: Differentiate given substitution.
- Step 2: Replace given variables (x and $\mathrm{d}x$).
- Step 3: Integrate.
- Step 4: Change back to the original variable.

Example

Use $u = x^2$ to find $\displaystyle\int \frac{x}{1-x^4}\,\mathrm{d}x$.

$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$$\begin{aligned}\int \frac{x}{1-x^4}\,\mathrm{d}x &= \int \frac{x}{1-u^2}\cdot\frac{1}{2x}\,\mathrm{d}u\\ &= \frac{1}{2}\int \frac{1}{1-u^2}\,\mathrm{d}u\\ &= \frac{1}{4}\ln\left\lvert\frac{1+u}{1-u}\right\rvert + C\\ &= \frac{1}{4}\ln\left\lvert\frac{1+x^2}{1-x^2}\right\rvert + C\end{aligned}$$

9. Definite integrals

The modulus

Approach

- For $|f(x)|$, figure out when the region when $f(x)$ is positive/negative (by drawing graphs or otherwise).
- Split up the integral. $|f(x)| = f(x)$ for regions where $f(x)$ is positive, and $|f(x)| = -f(x)$ otherwise.

Example

Evaluate

$$\int_0^5 |x^2 - 4| \, dx$$

$$\int_0^5 |x^2-4| \, dx = \int_0^2 -(x^2-4) \, dx + \int_2^5 (x^2-4) \, dx$$
$$= \left[-\frac{x^3}{3} + 4x\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_2^5$$
$$= \frac{97}{3}$$

Area under curves with parametric equations

Approach

- For area with respect to x-axis, we have $\boxed{\int_{x_1}^{x_2} y \, dx}$.
- For area with respect to y-axis, we have $\boxed{\int_{y_1}^{y_2} x \, dy}$.
- Similar to integration by substitution, we
  - Differentiate to get $\frac{dx}{dt}$ or $\frac{dy}{dt}$ to replace dx or dy.
  - Substitute x or y in terms of t.
  - Change the limits from x or y values to t values.
  - Evaluate the integral.

Volumes

Formula

- Rotation about x-axis:

$$\boxed{\pi \int_{x_1}^{x_2} y^2 \, dx}$$

- Rotation about y-axis:

$$\boxed{\pi \int_{y_1}^{y_2} x^2 \, dy}$$

Examples

Parametric equations

A curve C is defined by $x = t^2 + t, y = t - t^3$ for $t \geq 0$.

Find the area bounded by C and the x-axis.

$x = t^2 + t \Rightarrow \frac{dx}{dt} = 2t + 1$.
When $y = 0$,
$t = 0, x = 0$ or $t = 1, x = 2$.

$$\text{Area} = \int_0^2 y \, dx$$
$$= \int_0^1 (t - t^3)(2t+1) \, dt = \frac{31}{60}$$

Areas and volumes 1

The region R is bounded by the curve $y = x^2$, the line $x = 2$ and the x-axis.

Find the area of R, and the volumes when R is rotated about the x- and y-axes.

Area of $R = \int_0^2 x^2 \, dx$

Volume when R rotated about the x-axis
$= \pi \int_0^2 (x^2)^2 \, dx$

Volume when R rotated about the y-axis
$= \text{cylinder} - \pi \int_0^4 y \, dy$

Areas and volumes 2

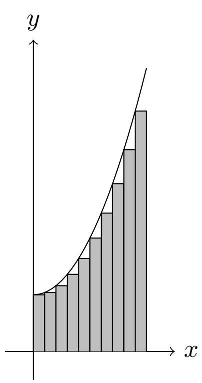
The region S is bounded by the curve $y = x^2$, the line $y = 4$ and the y-axis.

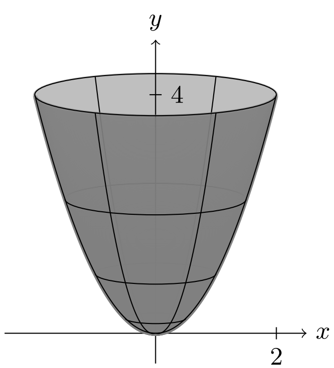
Find the area of S, and the volumes when S is rotated about the x- and y-axes.

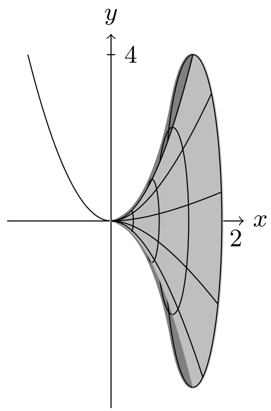
Handling limits, Riemann sums

- $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$.
- $\int_a^b u \frac{dv}{dx} \, dx = \Big[uv\Big]_a^b - \int_a^b \frac{du}{dx} v \, dx$.
- Remember to **change limits** when applying integration by substitution.

$$\int_0^1 f(x) \, dx = \lim_{n \to \infty} \frac{1}{n} \left(f(\frac{0}{n}) + f(\frac{1}{n}) + \ldots + f(\frac{n-1}{n}) \right)$$







Examples (continued)

Area of $S = \int_0^4 \sqrt{y} \, dy$

Volume when S rotated about the x-axis
$= \text{cylinder} - \pi \int_0^2 (x^2)^2 \, dx$

Volume when S rotated about the y-axis
$= \pi \int_0^4 y \, dy$

10. Differential equations

Theory

Theory

- **Differential equations** are equations involving variables (e.g. x, y) and their derivatives (e.g. $\frac{dy}{dx}$).
- They can be solved (i.e. finding an equation between x and y only) by integration.
- A differential equation is of **variable separable** form if it can be written as

$$g(y)\frac{dy}{dx} = f(x).$$

- Variable separable equations can be solved by proceeding to integrate:

$$\int g(y)\ dy = \int f(x)\ dx.$$

- Differential equations of the form $\frac{d^2y}{dx^2} = f(x)$ can be solved by integrating twice.
- The **general solution** refers to all possible solutions of a differential equation. There are infinite number of solutions since the integration constant C can take any value.
- A **particular solution** refers to one solution of a differential equation. Typically a set of values will be given to us to find C in order to obtain a particular solution.

Problem sums: rates of change

Rate of change of x: $\frac{dx}{dt}$

$$\boxed{\frac{dx}{dt} = \text{rate of increase of } x - \text{rate of decrease of } x}$$

Second order DE

$$\frac{d^2y}{dx^2} = 2x$$
$$\frac{dy}{dx} = x^2 + C$$
$$y = \frac{x^3}{3} + Cx + D.$$

Removing the modulus

$$\begin{aligned} \ln|x| &= t + C \\ |x| &= e^{t+C} \\ |x| &= Be^t && \text{where } B = e^C \\ x &= Ae^t && \text{where } A = \pm B \end{aligned}$$

Substitution

Approach

- Step 1: Differentiate given substitution implicitly.
- Step 2: Using the given substitution and the differentials in step 1, replace the old variable (e.g. y) to the new one (e.g. u).
- Step 3: The DE in the new variable should be of separable form. Integrate it.
- Step 4: Substitute back the old variable.

Variable separable form

$$\frac{dy}{dx} = xy^2$$
$$\frac{1}{y^2}\frac{dy}{dx} = x$$
$$\int \frac{1}{y^2}\ dy = \int x\ dx$$

General solution: $-\frac{1}{y} = \frac{x^2}{2} + C.$

If $y = 1$ when $x = 0$, $C = -1$

Particular solution: $-\frac{1}{y} = \frac{x^2}{2} - 1.$

Example: substitution method

Example

Use the substitution $u = xy$ to solve

$$x^2y\frac{dy}{dx} + xy^2 = 1$$

Differentiating $u = xy$ implicitly with respect to x,

$$\frac{du}{dx} = x\frac{dy}{dx} + y$$

Hence $x\frac{dy}{dx} = \frac{du}{dx} - y$

Substituting into given DE,

$$xy\left(\frac{du}{dx} - y\right) + uy = 1$$

$$u\frac{du}{dx} = 1$$
$$\frac{u^2}{2} = x + C$$
$$u^2 = 2x + C'$$

Solution: $(xy)^2 = 2x + C'$

11. Vectors I: basics, dot and cross products

Basics

Theory

- Two vectors $\mathbf{a}$ and $\mathbf{b}$ are **parallel** if $\boxed{\mathbf{a} = k\mathbf{b}}$ for some $k \neq 0$.
- The **position vector** of a point A with coordinates $(2, 3, -4)$ is the vector $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$.
- $\boxed{\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}}$
- Three points A, B and C are **collinear** if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \neq 0$.
- The **magnitude** of a vector is given by $\boxed{|a\mathbf{i} + b\mathbf{j} + c\mathbf{k}| = \sqrt{a^2 + b^2 + c^2}}$
- A vector $\mathbf{b}$ is a **unit vector** if $|\mathbf{b}| = 1$.
  A unit vector parallel to vector $\mathbf{a}$, denoted by $\hat{\mathbf{a}}$ can be calculated by $\boxed{\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}}$
- If C is between A and B such that $AC : CB = \lambda : \mu$, the **ratio theorem** gives us

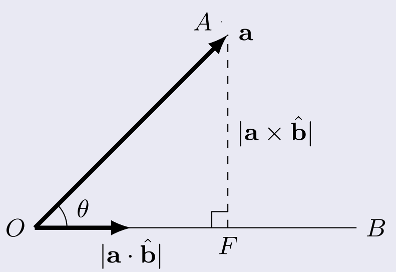
$$\boxed{\overrightarrow{OC} = \frac{\lambda\overrightarrow{OB} + \mu\overrightarrow{OA}}{\lambda + \mu}}$$

Dot and cross products

Theory

- The **dot/scalar** product: $\boxed{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad + be + cf}$
- Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. $\boxed{\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta}$
- If $\boxed{\mathbf{a} \cdot \mathbf{b} = 0}$, then $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ is **perpendicular** to $\mathbf{b}$.
- The **cross/vector** product: $\boxed{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} bf - ce \\ -(af - cd) \\ ae - bd \end{pmatrix}}$
- Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$
- If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, then $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ is parallel to $\mathbf{b}$.
- If $\mathbf{n} = \mathbf{a} \times \mathbf{b}$, then $\mathbf{n}$ is **perpendicular to both a and b**.
  We call $\mathbf{n}$ the **normal vector**.
- Area of triangle ABC: $\boxed{\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|}$.

Vector projection



- **Length of projection** of $\mathbf{a}$ onto $\mathbf{b}$ $\boxed{OF = \left|\mathbf{a} \cdot \hat{\mathbf{b}}\right|}$
- **Projection vector** $\boxed{\overrightarrow{OF} = \left(\mathbf{a} \cdot \hat{\mathbf{b}}\right)\hat{\mathbf{b}}}$
- **Perpendicular length** from A to OB: $\boxed{\left|\mathbf{a} \times \hat{\mathbf{b}}\right|}$

Vector algebra

$k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b})$	$k(\mathbf{a} \times \mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b})$
$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$	$\mathbf{a} \times \mathbf{a} = \mathbf{0}$
$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$	$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

Direction cosines

The **direction cosines** of a vector $\mathbf{d}$ are the angles between $\mathbf{d}$ and the x, y and z axes respectively.
$\alpha = \frac{d_1}{\sqrt{d_1^2+d_2^2+d_3^2}}$, $\beta = \frac{d_2}{\sqrt{d_1^2+d_2^2+d_3^2}}$, $\gamma = \frac{d_3}{\sqrt{d_1^2+d_2^2+d_3^2}}$.
$\alpha^2 + \beta^2 + \gamma^2 = 1$.

Equation of a line

Formula

Vector form:
$\boxed{l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}, \quad \lambda \in \mathbb{R}}$

Cartesian form:

$$\boxed{l : \frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{x - a_3}{d_3}}$$

$\mathbf{a}$: position vector of a **point** on the line

$\mathbf{d}$: **direction vector** parallel to the line

Equation of a plane

Formula

Vector/parametric form:
$\boxed{p : \mathbf{r} = \mathbf{a} + \lambda\mathbf{d_1} + \mu\mathbf{d_2}, \quad \lambda, \mu \in \mathbb{R}}$

Scalar product form: $\boxed{p : \mathbf{r} \cdot \mathbf{n} = K}$

Cartesian form:
$\boxed{p : n_1x + n_2y + n_3z = K}$

$\mathbf{a}$: position vector of a **point** on the plane

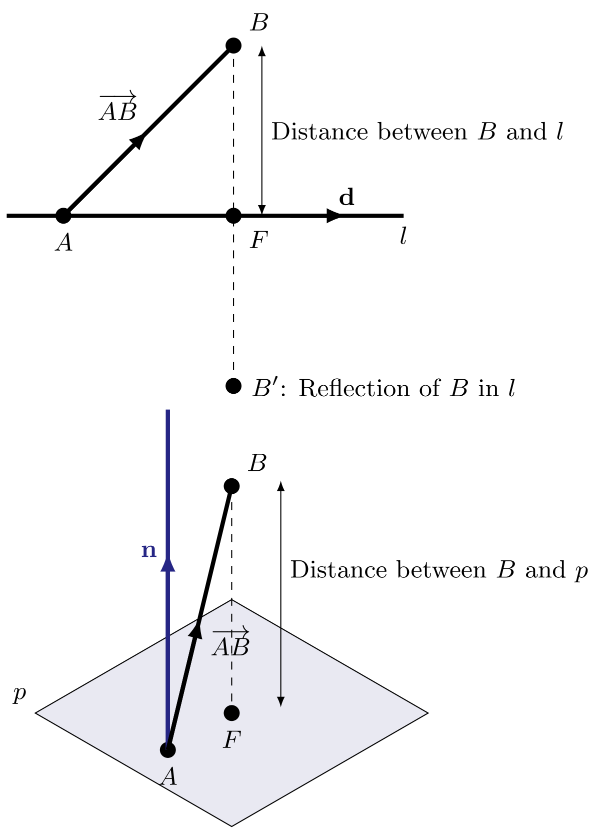
$\mathbf{d_1}, \mathbf{d_2}$: **direction vectors** parallel to the plane

$\mathbf{n}$: **normal vector** perpendicular to the plane

$K = \mathbf{a} \cdot \mathbf{n}$

12. Vectors II: lines and planes

Diagram



Example

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$$

$A(1, 2, -3)$
$B(3, -6, -2)$

$$l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

$p_2 : 5x + 3y + 2z = 0$

$$p_1 : \mathbf{r} \cdot \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = -5$$

Distances

Formula

- Distance between B and l: $\boxed{\left|\overrightarrow{AB} \times \hat{\mathbf{d}}\right|}$.
- Distance between B and p: $\boxed{\left|\overrightarrow{AB} \cdot \hat{\mathbf{n}}\right|}$.

Foot of perpendiculars

For foot of perpendicular of B on l_1:

Let $\overrightarrow{OF} = \begin{pmatrix} 1+\lambda \\ 2-3\lambda \\ -3+2\lambda \end{pmatrix}$ since F lies on l_1.

$\overrightarrow{BF} = \overrightarrow{OF} - \overrightarrow{OB}$.
Since $\overrightarrow{BF} \perp \mathbf{d}$, $\overrightarrow{BF} \cdot \mathbf{d} = 0$.
Solve for λ to obtain $\overrightarrow{OF}$.

For foot of perpendicular of B on p_1:

Equation of line $BF : \mathbf{r} = \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix} + \nu \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \nu \in \mathbb{R}$.

Since F is the intersection point between BF and p:

$$\begin{pmatrix} 3 \\ -6+2\nu \\ -2+3\mu \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = -5.$$

Solve for ν to obtain $\overrightarrow{OF}$.

For point of reflection, B', of B in l_1 or p_1,

by ratio theorem, $\boxed{\overrightarrow{OF} = \dfrac{\overrightarrow{OB} + \overrightarrow{OB'}}{2}}$.

Skew lines

Two lines are skew if

- they are not parallel: $\mathbf{d_1} \neq k\mathbf{d_2}$
- they do not intersect: solving their equations simultaneously does not yield unique solutions.

Angles

Formula

- Angle between l_1 and l_2: $\boxed{\mathbf{d_1} \cdot \mathbf{d_2} = |\mathbf{d_1}||\mathbf{d_2}| \cos\theta}$.
- Angle between l and p: $\boxed{\mathbf{d} \cdot \mathbf{n} = |\mathbf{d}||\mathbf{n}| \sin\theta}$.
- Angle between p_1 and p_2: $\boxed{\mathbf{n_1} \cdot \mathbf{n_2} = |\mathbf{n_1}||\mathbf{n_2}| \cos\theta}$.

Intersections

For point of intersection between l_1 and l_2:

$$\begin{pmatrix} 1+\lambda \\ 2-3\lambda \\ -3+2\lambda \end{pmatrix} = \begin{pmatrix} 2-\mu \\ -3+\mu \\ 2+\mu \end{pmatrix}.$$

Solve for λ and/or μ.

For point of intersection between l_1 and p_1:

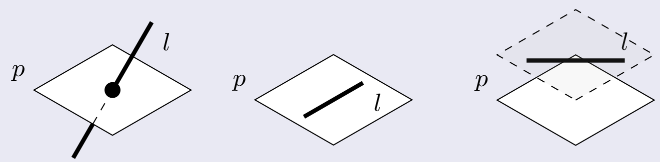
$$\begin{pmatrix} 1+\lambda \\ 2-3\lambda \\ -3+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = -5.$$

Solve for λ.

For line of intersection between p_1 and p_2,
we convert both planes to cartesian form:
$p_1 : 2y + 3z = -5$
$p_2 : 5x + 3y + 2z = 0$.
We solve both equations simultaneously in our GC to get

$$l : \mathbf{r} = \begin{pmatrix} 1.5 \\ -2.5 \\ 0 \end{pmatrix} + \omega \begin{pmatrix} 0.5 \\ -1.5 \\ 1 \end{pmatrix}.$$

Line and plane



(a) $\mathbf{d} \cdot \mathbf{n} \neq 0$
Intersect at 1 point

(b) $\mathbf{d} \cdot \mathbf{n} = 0$,
$\mathbf{a} \cdot \mathbf{n} = K$
Line lies in plane

(c) $\mathbf{d} \cdot \mathbf{n} = 0$,
$\mathbf{a} \cdot \mathbf{n} \neq K$
Line parallel to plane

13. Complex numbers

Basics, complex conjugates

Theory

- A complex number z is of the form $\boxed{z = x + yi}$, where $x, y \in \mathbb{R}$ and $i^2 = -1$.
- We call x the **real part** $\mathrm{Re}(z) = x$ and y the **imaginary part** $\mathrm{Im}(z) = y$.
- The **complex conjugate** of $z = x + yi$ is given by $\boxed{z^* = x - yi}$.
  - $\boxed{z + z^* = 2x = 2\mathrm{Re}(z)}$
  - $\boxed{z - z^* = 2yi = 2i\mathrm{Im}(z)}$
  - $\boxed{zz^* = x^2 + y^2 = |z|^2}$

Example of complex division:

$$\frac{1-i}{3+4i} = \frac{1-i}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{-1-7i}{25}$$

The quadratic formula

Example

Solve $z^2 + 2z + 5 = 0$.

$$z = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

Modulus/argument I

Formula

- $\boxed{r = |z| = \sqrt{x^2 + y^2}}$
- Let $\arg(z) = \theta$. $\boxed{\tan\theta = \frac{y}{x}}$

Comparing parts

Example

Solve $z^2 + zz^* = 8 - 4i$.

Let $z = x + yi$

$(x + yi)^2 + (x + yi)(x - yi) = 8 - 4i$

$x^2 + 2xyi - y^2 + x^2 + y^2 = 8 - 4i$

$2x^2 + 2xyi = 8 - 4i$

Comparing real parts:

$2x^2 = 8 \;\Rightarrow\; x = \pm 2$

Comparing imaginary parts:

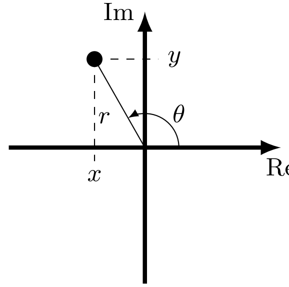
$2xy = -4 \;\Rightarrow\; y = \mp 1$

Hence $z = 2 - i$ or $z = -2 + i$.

Modulus/argument II

Let $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$$\theta = \begin{cases} \alpha & \text{first quadrant} \\ \pi - \alpha & \text{second quadrant} \\ -(\pi - \alpha) & \text{third quadrant} \\ -\alpha & \text{fourth quadrant} \end{cases}$$



The Argand diagram

Complex number forms

Formula

- **Cartesian form**: $\boxed{z = x + yi}$
- **Polar/trigo form**: $\boxed{z = r(\cos\theta + i\sin\theta)}$
- **Euler/exp form**: $\boxed{z = re^{i\theta}}$

The conjugate root theorem

Theory

- Let $P(z)$ be a polynomial with **real coefficients**. The **conjugate root theorem** states that if $a + bi$ is a root to $P(z) = 0$, then its conjugate $a - bi$ is also a root to $P(z) = 0$.

Example

Solve $3z^3 - 7z^2 + 17z - 5 = 0$ given that $1 + 2i$ is a root.

Since all the coefficients are real, by the conjugate root theorem, $1 - 2i$ is also a root.

By the **factor theorem**, $(z - (1 + 2i))$ and $(z - (1 - 2i))$ are factors of the cubic polynomial.

$(z - 1 - 2i)(z - 1 + 2i) = (z - 1)^2 - (2i)^2 = z^2 - 2z + 5$.

By **long division** or **comparing coefficients**, we can obtain the final factor $3z - 1$.

$3z^3 - 7z^2 + 17z - 5 =$
$(z - (1 + 2i)(z - (1 - 2i))(3z - 1)$.

Hence $z = 1 + 2i, z = 1 - 2i$ or $z = \frac{1}{3}$.

Modulus/argument III

$$|wz| = |w||z| \qquad \arg(wz) = \arg(w) + \arg(z)$$
$$\left|\frac{w}{z}\right| = \frac{|w|}{|z|} \qquad \arg\left(\frac{w}{z}\right) = \arg(w) - \arg(z)$$
$$|z^n| = |z|^n \qquad \arg(z^n) = n\arg(z)$$
$$|z^*| = |z| \qquad \arg(z^*) = -\arg(z)$$

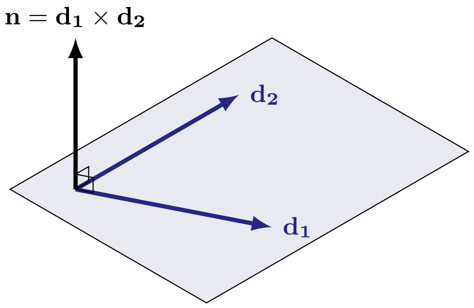
A special example

$$1 + e^{i2\theta} = e^{i\theta}e^{-i\theta} + e^{i\theta}e^{i\theta} = e^{i\theta}\left(e^{-i\theta} + e^{i\theta}\right) = e^{i\theta}(2\mathrm{Re}(e^{i\theta})) = 2\cos\theta\, e^{i\theta}$$

Purely real/imaginary numbers

Condition	Cartesian	Argument ($k \in \mathbb{Z}$)
real	$y = 0$	$\arg = k\pi$
real and positive	$y = 0,\ x > 0$	$\arg = 2k\pi$
real and negative	$y = 0,\ x < 0$	$\arg = (2k+1)\pi$
purely imaginary	$x = 0$	$\arg = \frac{(2k+1)\pi}{2}$

Contents



Author's note

This set of notes presents a brief summary of the terms, concepts and common techniques found in the H2 A Level Mathematics (9758) syllabus. It is intended as a handy companion for students by laying out the key content, with every chapter summarized into one page. It may also be useful for students taking other pre-university mathematics courses.

In preparing this set of notes we have strived to be comprehensive and rigorous, yet at the same time clear and concise. Much thought has gone into which items to include or exclude, and when to use general cases vs specific examples. Trying to fit each topic into a single page have added to the challenge, but we think the benefit of having a handy resource outweighs the downside of the format. We hope we have not erred too much on our choices.

To our readers: we look forward to the corrections of our inevitable mistakes and any comments or suggestions about these notes. We have found much joy and meaning in our (ongoing) mathematical journey and hope you will find yours as rewarding.

Dedicated to all my teachers and students.

Useful links

- `https://www.seab.gov.sg/home/examinations/gce-a-level/`: A Level syllabus and formula List MF26.
- `https://www.desmos.com/`: a useful online graphing calculator.
- `http://www.adotb.xyz/`: the author's website.

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Contact me at KELVINSJK (AT) GMAIL.COM.