MERIDIAN JUNIOR COLLEGE
JC2 Preliminary Examination
Higher 1

H1 Mathematics 8865/01

Paper 1 **14 September 2018**

3 Hours

Additional Materials: Writing paper
Graph Paper
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

[Turn Over

Section A: Pure Mathematics [40 marks]

1 **(a)** C_1 and C_2 have equations $y = \mathrm{e}^{3-2x}$ and $y = 5\mathrm{e}^{2x}$ respectively. Determine the exact coordinates of the intersection point. [5]

(b) The graph of $y = ax^2 + bx + c$ has turning point $(1, 4)$ and y-intercept $(0, -1)$.

Determine the values of a, b and c. [4]

2 **(a)** Find the range of values of p for which the equation $x^2 + px + 2 = 0$ has no real roots. [2]

(b) Find the range of values of k for the line $y = 2x + 3$ to intersect the curve $y = kx^2 + (2k+1)x - 3$ at least once. [4]

3 **(a)** The gradient at any point on a particular curve is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = (2-x)^2$. Given that the curve passes through the point $(0, 2)$, find the equation of the curve. [3]

Hence, find the area of the region bounded by the curve, the x- and y-axes and the line $x = 5$. [1]

(b) Evaluate, in terms of k, the integral $\int_2^k \frac{1}{x^2} - \frac{2}{1-x} \, \mathrm{d}x$ where $k > 2$. [3]

4 [It is given that the volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.]

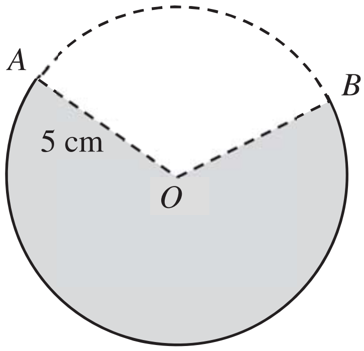


Figure 1

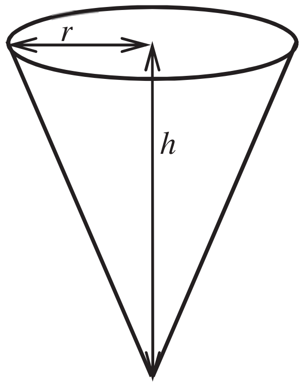


Figure 2

Figure 1 shows a circular piece of paper of radius 5 cm after removing the minor sector OAB. The edges OA and OB are joined to form a cone-shaped drinking cup with radius r and height h as shown in Figure 2.

(i) Show that the volume, V, of the conical cup can be expressed as $\frac{1}{3}\pi\left(25h - h^3\right)$. [2]

(ii) Using differentiation, find the maximum volume of the cup. [5]

(iii) After the cup is fully filled, water leaks at a rate of 3 cm^3 per minute from the cup. Find the exact rate of change of the height of the water in the cup when the height of water is 2 cm. [3]

5 **(i)** Sketch the graph of C with equation $y = \ln\left(2x+3\right)$, stating the equation of any asymptotes and the exact coordinates of intersections with the axes. [2]

The point P on C has coordinates $\left(1, \ln 5\right)$. The tangent to C at P meets the x-axis at T.

(ii) Show that the exact x-coordinate of T is $1 - \frac{5}{2}\ln 5$. [4]

(iii) Hence find the area bounded by the tangent to C at P, the x-axis and the curve C, leaving your answers to 4 decimal places. [2]

Section B: Statistics [60 marks]

6 The seven letters in the word ELEMENT are to be arranged to form different 7-letter code words. Find the probability that a code word chosen at random

(i) has all the three E's next to each other, [2]

(ii) has all the three E's separated. [3]

7 An egg wholesaler packs their chicken eggs according to their weights. The chicken eggs are weighed and classified according to the table below.

Criteria	Classification of egg
Weight of chicken egg more than 65 grams	Premium
Weight of chicken egg less than 48 grams	Small

A large batch of randomly chosen chicken eggs are weighed and classified accordingly and it is found that 12% are premium and 6% are small. Assuming a normal distribution, find the mean weight and standard deviation of a randomly chosen chicken egg. [5]

The wholesaler also distributes ostrich eggs for sale.

Explain whether or not a normal model is likely to be appropriate for the weight of an egg chosen at random from the combined group of chicken eggs and ostrich eggs. [1]

8 The events A and B are such that $\mathrm{P}(A') = 0.14$, $\mathrm{P}(B) = 0.34$ and $\mathrm{P}(A' \cap B') = 0.08$.

(i) Draw a Venn Diagram to represent this situation, showing the probability in each of the four regions. [3]

(ii) Determine if A and B are independent events. [2]

(iii) Find the probability that exactly one of A and B occurs. [1]

(iv) Given that $\mathrm{P}(C \mid A) = p$ and events B and C are mutually exclusive, find the largest possible value of p. [3]

9 A machine produces n toys a day. Over a long time, it is found that $100p\%$ of the toys produced by the machine is defective. The number of defective toys produced in a day by the machine is denoted by X.

(i) State, in the context of this question, two conditions needed for X to be well modelled by a binomial distribution. [2]

Assume now that X indeed follows a binomial distribution.

(ii) The mean number and variance of defective toys produced by the machine in a day is 10 and 9.75 respectively. Find the value of n and p. [3]

(iii) Given that there are at most five defective toys produced by the machine on a particular day, find the probability that the last toy produced by the machine is defective. [4]

10 Siti runs a full marathon every year at the annual Scanchart marathon since 2010 and she records the time she takes to complete the marathon, in minutes above 3 hours 30 minutes, starting from her first year of participation. The results are shown in the table below.

Year (x)	2010	2011	2012	2013	2014	2015	2016	2017
time (t)	80.2	56.6	48.2	42.5	37.3	34.9	29.7	25.0

(i) Draw a scatter diagram for the above data, clearly labelling the axes. [2]

(ii) Find the product moment correlation coefficient and comment on its value in the context of the question. [2]

(iii) Find the equation of the regression line t on x. Interpret the gradient of the regression line in the context of the question. [2]

(iv) Using the regression line you have found in **(iii)**, predict Siti's timing for the marathon she will run in 2018. Comment on the reliability of your prediction. [2]

(v) Explain why a linear model may not be appropriate in this context. [1]

11 A manufacturer claims that his company produces cans of coffee that have an average volume of 230 millilitres each, with a standard deviation of 10 millilitres. A random sample of 80 such cans is taken. A test, is carried out at the 5% significance level, on whether the manufacturer has overstated the mean volume.

(i) Write down appropriate hypotheses to test the manufacturer's claim, defining any symbols you use. [2]

(ii) State what you understand by the term "at 5% level of significance" in the context of the question. [1]

(iii) Use an algebraic method to calculate the set of possible values of the average volume of beverage dispensed for which the null hypothesis would not be rejected. [5]

(iv) Explain why there is no need to know anything about the population distribution of the volumes of the beverages. [2]

12 Brandon can choose to take the subway or taxi to and fro between his home and office. The one-way journey times, in minutes, by taxi and by subway have independent normal distributions. The means and standard deviations of these distributions are shown in the following table.

	Mean	Standard Deviation
Taxi	59	2
Subway	61	3

(i) Find the probability that a randomly chosen taxi journey takes less than an hour. [1]

(ii) Find the probability that two randomly chosen taxi journeys take more than an hour each. [2]

(iii) The probability that the total journey time taken for a randomly chosen trip to and fro between home and office by taxi is more than two hours in total is denoted by p. Without calculating its value, explain why p will be greater than your answer in part **(ii)**. [1]

(iv) Brandon takes a subway from his home to his office to pick up a bulky item. He then takes a taxi back to his home to store the item. Finally, he takes a taxi back to his office. Find the probability that Brandon's total journey takes more than 3 hours. [3]

Journeys are charged by the time taken. For the taxi journey, the charge is \$0.69 per minute and for the subway journey, the charge is \$0.13 per minute.

Let A represent the cost of the taxi journey from Brandon's home to work.

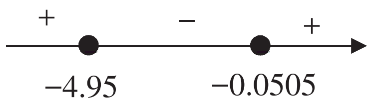
Let B represent the cost of the subway journey from Brandon's home to work.

(v) Find $\mathrm{P}\left(2A-\left(B_1+B_2\right)<68\right)$ and explain, in the context of the question, what your answer represents. [5]

End of Paper

2018 H1 MATH (8865/01) JC 2 PRELIM – SUGGESTED SOLUTIONS

Qn	Solutions
1	**Exponential & Logarithm**
(a)	$\mathrm{e}^{3-2x}=5\mathrm{e}^{2x}$ $\ln\left(\mathrm{e}^{3-2x}\right)=\ln 5+\ln\left(\mathrm{e}^{2x}\right)$ $\Rightarrow 3-2x=\ln 5+2x$ $\Rightarrow x=\dfrac{3-\ln 5}{4}=\dfrac{3}{4}-\dfrac{1}{4}\ln 5$ $\Rightarrow y=5\mathrm{e}^{2\left(\frac{3-\ln 5}{4}\right)}$ $\Rightarrow y=5\mathrm{e}^{\left(\frac{3-\ln 5}{2}\right)}$ Hence coordinates are $\left(\dfrac{3}{4}-\dfrac{1}{4}\ln 5,\ \ 5\mathrm{e}^{\left(\frac{3-\ln 5}{2}\right)}\right)$.
(b)	$y=ax^2+bx+c$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2ax+b$ At $(1,4)$, $4=a+b+c$ $0=2a+b$ At $(0,-1)$, $-1=c$ Using GC to solve, $a=-5,\ b=10$.

Qn	Solution
2	**Equations & Inequalities**
(a)	$x^2 + px + 2 = 0$ No real roots $\Rightarrow$ Discriminant < 0 $p^2 - 4(1)(2) < 0$ $\left(p - \sqrt{8}\right)\left(p + \sqrt{8}\right) < 0$ $-\sqrt{8} < p < \sqrt{8}$ $-2\sqrt{2} < p < 2\sqrt{2}$
(b)	$kx^2 + (2k+1)x - 3 = 2x + 3$ $kx^2 + (2k+1)x - 3 - 2x - 3 = 0$ $kx^2 + (2k-1)x - 6 = 0$ Intersect at least once $\Rightarrow$ Discriminant ≥ 0 $(2k-1)^2 - 4k(-6) \geq 0$ $4k^2 - 4k + 1 + 24k \geq 0$ $4k^2 + 20k + 1 \geq 0$  Using GC, $k \leq -4.95$ or $k \geq -0.0505$ (3 s.f.)

Qn	Solution
3	**Techniques of Integration**
(a)	$\frac{dy}{dx} = (2-x)^2$ Equation of curve is $y = \int (2-x)^2 dx = \int x^2 - 4x + 4\ dx$ $y = \frac{x^3}{3} - 2x^2 + 4x + C$ Since (0,2) is on the curve, $C = 2$. $\therefore y = \frac{x^3}{3} - 2x^2 + 4x + 2$ Using GC, required area = $\int_0^5 \frac{x^3}{3} - 2x^2 + 4x + 2\ dx = 28.75\ \text{units}^2$
(b)	$\int_2^k \frac{1}{x^2} - \frac{2}{1-x}\ dx$ $= \left[-\frac{1}{x} + 2\ln\lvert 1-x \rvert \right]_2^k$ $= \left(-\frac{1}{k} + 2\ln\lvert 1-k \rvert \right) - \left(-\frac{1}{2} + 2\ln\lvert 1-2 \rvert \right)$ $= -\frac{1}{k} + 2\ln\lvert 1-k \rvert + \frac{1}{2}$

Qn	Solution
4	**Application of Differentiation**

(i)

Using Pythagoras' Theorem,

$$r^2 + h^2 = 5^2$$

$$\Rightarrow r^2 = 25 - h^2$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(25 - h^2\right)h$$

$$= \frac{1}{3}\pi\left(25h - h^3\right)$$

(ii)

$$V = \frac{1}{3}\pi\left(25h - h^3\right)$$

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{3}\pi\left(25 - 3h^2\right)$$

For max vol, $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$

$$\frac{1}{3}\pi\left(25 - 3h^2\right) = 0$$

$$25 - 3h^2 = 0$$

$$h^2 = \frac{25}{3}$$

since $h > 0, h = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$

1st derivative test

h	$\left(\frac{5\sqrt{3}}{3}\right)^-$	$\frac{5\sqrt{3}}{3}$	$\left(\frac{5\sqrt{3}}{3}\right)^+$
$\frac{\mathrm{dV}}{\mathrm{dh}}$	+	0	-
Slope of curve	/	—	\

$\therefore h = \frac{5\sqrt{3}}{3}$ gives max vol.

Alternatively: Using 2nd derivative test

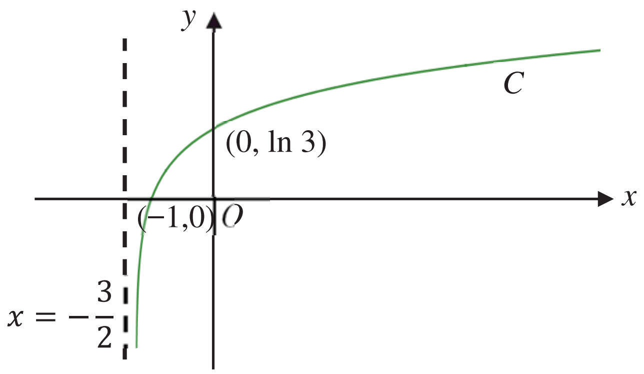
$$\frac{\mathrm{d}^2V}{\mathrm{d}h^2} = \frac{1}{3}\pi\left(-6h\right)$$

$$= -2\pi h$$

When $h = \frac{5\sqrt{3}}{3}, \frac{\mathrm{d}^2V}{\mathrm{d}h^2} = -\frac{10\sqrt{3}\pi}{3} < 0$

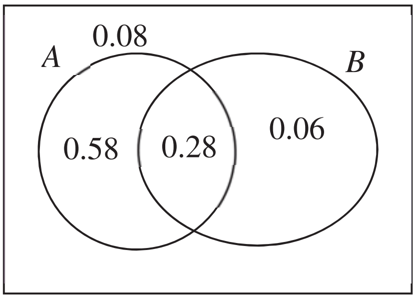
$\therefore h = \frac{5\sqrt{3}}{3}$ gives max vol.

	$V=\frac{1}{3}\pi\left(25\left(\frac{5\sqrt{3}}{3}\right)-\left(\frac{5\sqrt{3}}{3}\right)^3\right)$ $=50.3833$ $=50.4$ (to 3 s.f.)
(iii)	$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ when $h=2, \frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{3}\pi\left(25-3\left(2^2\right)\right)=\frac{13\pi}{3}$ Given $\frac{\mathrm{d}V}{\mathrm{d}t}=-3$, $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{3}{13\pi}\times(-3)$ $=\frac{-9}{13\pi}$

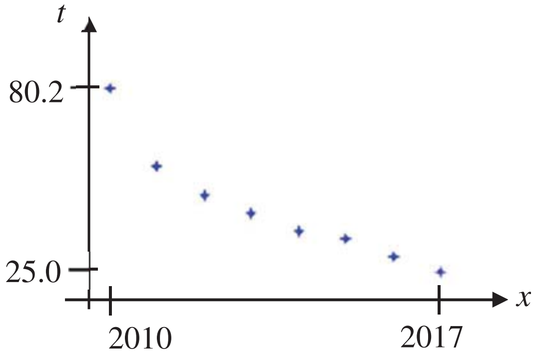
Qn	Solution
5	**Applications of Differentiation & Integration & Curve Sketching**
(i)	
(ii)	$y=\ln(2x+3)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{2x+3}$ At $x=1$, gradient of tangent = $\frac{2}{5}$ Equation of tangent : $y-\ln 5=\frac{2}{5}(x-1)$ At T, $y=0$, $0-\ln 5=\frac{2}{5}(x-1)$ $-\frac{5}{2}\ln 5=x-1$ $x=1-\frac{5}{2}\ln 5$ (Shown)
(iii)	Area = $\int_{1-\frac{5}{2}\ln 5}^{1}\left(\frac{2}{5}(x-1)+\ln 5\right)\mathrm{d}x-\int_{-1}^{1}\ln(2x+3)\,\mathrm{d}x=1.2143$(4 d.p.) **OR** Area = $\frac{1}{2}\left[1-\left(1-\frac{5}{2}\ln 5\right)\right](\ln 5)-\int_{-1}^{1}\ln(2x+3)\,\mathrm{d}x=1.2143$(4 d.p.)

Qn	Solution
6	**Probability (Using P&C)**
(i)	Total number of arrangements = $\frac{7!}{3!}=840$ Number of arrangements where all the three E's are next to each other = 5! Required probability = $\frac{5!}{840}=\frac{1}{7}$
(ii)	Number of ways to arrange L, M, N T = 4! = 24 Number of ways to slot in the three E's = $\binom{5}{3}=10$ Required probability = $\frac{24\times10}{840}=\frac{2}{7}$

Qn	Solutions
7	**Normal Distribution**
(i)	Let X kg be the weight of a randomly chosen egg. $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$ $\mathrm{P}(X > 65) = 0.12 \Rightarrow \mathrm{P}\left(Z > \frac{65-\mu}{\sigma}\right) = 0.12$ $\Rightarrow \frac{65-\mu}{\sigma} = 1.174987$ $65 - \mu = 1.174987\sigma$ -------- (1) $\mathrm{P}(X < 48) = 0.06 \Rightarrow \mathrm{P}\left(Z < \frac{48-\mu}{\sigma}\right) = 0.06$ $\Rightarrow \frac{48-\mu}{\sigma} = -1.554774$ $48 - \mu = -1.554774\sigma$ -------- (2) Solving using GC $\sigma = 6.2277 \approx 6.23$ $\mu = 57.683 \approx 57.7$
	A normal model is not appropriate because when we combine both groups with different distributions, it may result in a bi-modal distribution (i.e. with two modes) which contrasts with a normal distribution that has only one mode.

Qn	Solution
8	**Probability (Using Venn Diagrams)**
(i)	
(ii)	$\mathrm{P}(A)\mathrm{P}(B)=(0.86)(0.34)=0.2924\neq 0.28$ Since $\mathrm{P}(A\cap B)\neq \mathrm{P}(A)\mathrm{P}(B)$, A and B are not independent.
(iii)	Required probability = 0.58 + 0.06=0.64
(iv)	$\mathrm{P}(C\mid A)=p$ $\dfrac{\mathrm{P}(C\cap A)}{\mathrm{P}(A)}=p$ $\mathrm{P}(C\cap A)=0.86p$ Largest p occurs when $\mathrm{P}(C\cap A)=0.58$ $0.58 = 0.86p$ $p=\dfrac{29}{43}=0.674$ (3 s.f.)

Qn	Solution
9	**Binomial Distribution**
(i)	The probability that a randomly chosen toy is defective is constant for all toys. Whether a randomly chosen toy is defective is independent from any other toys.
(ii)	$X \sim \mathrm{B}(n, p)$ $\mathrm{E}(X) = 10 \quad \Rightarrow \quad np = 10$ $\mathrm{Var}(X) = 9.75 \quad \Rightarrow \quad np(1-p) = 9.75$ $1 - p = \dfrac{9.75}{10} \quad \Rightarrow \quad p = 0.025$ $\therefore n = \dfrac{10}{0.025} = 400$
(iii)	Let Y be the number of defective toys produced out of 399 toys $Y \sim \mathrm{B}(399, 0.025)$ Required probability $= \dfrac{\mathrm{P}(Y \le 4) \times 0.025}{\mathrm{P}(X \le 5)}$ = 0.0109 (3 s.f.)

Qn	Solution
10	**Correlation and Regression**
(i)	
(ii)	Using GC, $r = -0.93492 = -0.935$ (3s.f.) Since r is close to −1, there is a strong negative linear correlation between the year of participation, x and the time above 3 hours and 30 minutes, she takes to complete the marathon, t.
(iii)	Using GC, $t = 13611 - 6.7381x$ $\Rightarrow t = 13600 - 6.74x$ (to 3 s.f.) The gradient of −6.74 means that the time taken for Siti to complete the marathon decreases by 6.74 minutes with the passing of every year.
(iv)	When $x = 2018$, $t = 13611 - 6.7381(2018)$ $t = 13.978 = 14.0$ min (to 3 s.f) The predicted time for Siti to complete the marathon in 2018 is 3 hours 44 min. Since $x = 2018$ falls outside the data range of x, the linear relation between t and x may no longer hold. Thus the prediction is not reliable.
(v)	A linear model would not be appropriate in the long run since Siti will be limited by her human capacity and her timing at some point in time is likely to reach a plateau. (OR, a linear model would imply an indefinite decrease, which would imply a negative time in the long run which is not possible.)

Qn	Solution
11	**Hypothesis Testing**
(i)	Let X be the volume of beverage dispensed by the vending machine (in millilitres). Let μ denote the population mean volume of beverage dispensed by the vending machine (in millilitres). $\text{H}_0 : \mu = 230$ $\text{H}_1 : \mu < 230$
(ii)	There is a probability of 0.05 of incorrectly concluding that the volume of the can of coffee is less than 230 millilitres of beverage, when in fact, it is 230 millilitres.
(iii)	Under H_0, since n = 80 is large, by Central Limit Theorem, $\overline{X} \sim \text{N}\left(230, \frac{10^2}{80}\right)$ approximately. Test Statistic: $Z = \dfrac{\overline{X} - 230}{\sqrt{\dfrac{10^2}{80}}}$ Level of significance: 5% Reject H_0 if z-value < -1.6449 Under H_0, $z-\text{value} = \dfrac{\overline{x} - 230}{\sqrt{10^2/80}}$ Since the null hypothesis is not rejected at 5% significance level, H_0 should not be rejected. z-value > -1.6449 $\dfrac{\overline{x} - 230}{\sqrt{10^2/80}} > -1.6449$ $\overline{x} > 228.16$ $\therefore \ \{\overline{x} \in \mathbb{R}^+ : \overline{x} > 228\}$
(iv)	There is no need to know anything about the population distribution as the sample size, 80, is sufficiently large and thus by Central Limit Theorem, $\overline{X} \sim \text{N}\left(230, \frac{10^2}{80}\right)$ approximately.

Qn	Solution
12	**Sampling**
(i)	Let X and Y be the journey times by a randomly chosen taxi and a randomly chosen subway respectively (in minutes). $X \sim \mathrm{N}\left(59, 2^2\right)$ $Y \sim \mathrm{N}\left(61, 3^2\right)$ $\mathrm{P}(X < 60) = 0.69146 \approx 0.691$ (3s.f.)
(ii)	$\left[\mathrm{P}(X > 60)\right]^2 = (1 - 0.69146)^2 = 0.095195 \approx 0.0952$ (3s.f.)
(iii)	The event in **(ii)** is a proper subset of the event in **(iii)**, thus p is greater than the answer in part **(ii)**.
(iv)	$X_1 + X_2 + Y \sim \mathrm{N}\left(2\times 59 + 61, 2\times 2^2 + 3^2\right)$ $X_1 + X_2 + Y \sim \mathrm{N}(179, 17)$ $\mathrm{P}(X_1 + X_2 + Y_3 > 180) = 0.40418 \approx 0.404$ (3s.f.)
(v)	$A = 0.69X \sim \mathrm{N}(40.71, 1.9044)$ $B = 0.13Y \sim \mathrm{N}(7.93, 0.1521)$ $2A - (B_1 + B_2) \sim \mathrm{N}(65.56, 7.9218)$ $\mathrm{P}\left(2A - (B_1 + B_2) < 68\right) = 0.80701 \approx 0.807$ The answer represents the probability that 2 times the cost of a randomly chosen taxi journey exceeds the cost of the sum of 2 randomly chosen subway journeys by less than \$68.