H2

TAMPINES JUNIOR COLLEGE
JC2 SEMESTRAL ASSESSMENT

MATHEMATICS 9758/01

Paper 1 Wednesday, 27 June 2018

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in, including the Cover Page.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages.

1 With the aid of a sketch, solve the inequality

$$\frac{1}{2}x+8>|2x-4|.$$ [4]

2 Find the exact value of p such that

$$\int_1^3 \frac{6}{x^2+3}\,dx=\int_{e^{-2}}^{e^{-p}} \frac{1}{x\ln\left(\frac{1}{x}\right)}\,dx,$$

where $0<p<2$. [5]

3 The region bounded by the curve $y=\cos x$, the positive x-axis and positive y-axis is rotated through 2π radians about the y-axis. Use the substitution $y=\cos u$ to show that the volume of the solid obtained is given by

$$\pi\int_0^{\frac{\pi}{2}} u^2 \sin u\,du$$

and evaluate this integral exactly. [6]

4 The function f is given by $f: x \mapsto \frac{2-x^2}{1-x^2}$, for $x\in\mathbb{R}, x\geq 0, x\neq 1$.

(i) Show that f has an inverse. [2]

(ii) Find $f^{-1}(x)$ and state the domain of f^{-1}. [3]

(iii) Determine, with a reason, whether the composite function ff exists. Hence explain why there is no value of x which satisfies the equation $f(x)=f^{-1}(x)$. [2]

5 Show that $\frac{2}{4r^2-1}$ can be expressed as $\frac{A}{2r-1}+\frac{B}{2r+1}$, where A and B are constants to be determined. [2]

(i) Find $\sum_{r=1}^{n}\frac{1}{4r^2-1}$. [3]

(ii) Give a reason why the series $\sum_{r=1}^{\infty}\frac{1}{4r^2-1}$ converges, and write down its value. [2]

(iii) Find $\sum_{r=1}^{n}\frac{1}{4(r+2)^2-1}$. [4]

6 A curve C has parametric equations

$$x = \cos^3\theta, \qquad y = 6\cos^2\theta\sin\theta \qquad \text{for } 0 \le \theta \le \frac{\pi}{2}.$$

(i) Show that $\frac{dy}{dx} = -2\cot\theta + 4\tan\theta$. [3]

(ii) Find the equation of the tangent to C at the point where $\theta = \frac{\pi}{4}$. [2]

(iii) Point A is a stationary point on C. Show that the stationary point occurs when $\tan\theta = \frac{1}{\sqrt{a}}$, where a is an integer to be determined. Find the exact coordinates of A. [4]

(iv) The line with equation $y = bx$, where b is a positive constant, meets C at the origin and at the point B. Show that $\tan\theta = \frac{b}{6}$ at B. Find the exact value of b such that the line passes through A. [3]

7 **Do not use a calculator in answering this question.**

(i) Showing your working, find the complex numbers z and w which satisfy the simultaneous equations

$$(1-\mathrm{i})z + 4w = 8 \qquad \text{and}$$
$$-2\mathrm{i}z + (1-6\mathrm{i})w = -9\mathrm{i}.$$
[5]

(ii) Find an exact expression for z^7. Give your answer in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$. [3]

(iii) The complex conjugate of z is denoted by z^*. Find the three smallest positive integers of n for which $\frac{z^*}{z^n}$ is purely imaginary. [4]

8 There are two national saving plans available, Plan A and Plan B, both requiring monthly deposits of a fixed amount of money on the first day of each month. In Plan A, an interest rate of 0.45% per month is given on the last day of each month.

(i) Saver A chooses Plan A and deposits \$200 on the first day of each month. Show that the total savings for saver A at the end of n months is

$$\$\frac{401800}{9}(1.0045^n - 1).$$ [2]

(ii) Hence calculate the total interest saver A receives after 12 months. [2]

(iii) Saver A decides to withdraw the money once the total savings reaches \$4000. How many months must saver A wait in order to withdraw the money? [3]

Plan B gives a fixed interest of \$0.10 on the last day of each month. Saver B chooses Plan B and deposits \$$y$ on the first day of each month.

(iv) Find the least value of y (to nearest dollar) for saver B to deposit each month in order to save more than saver A at the end of 12 months, assuming both saver A and saver B start their saving plan at the same time. [5]

9 Two sloping roofs of an outdoor shelter can be modelled by two planes p_1 and p_2 with equations $3x + y + 2z = 5$ and $x - 2y + 3z = 4$ respectively. Points (x, y, z) are defined relative to the base of a supporting pillar at $(0,0,0)$ where units are metres.

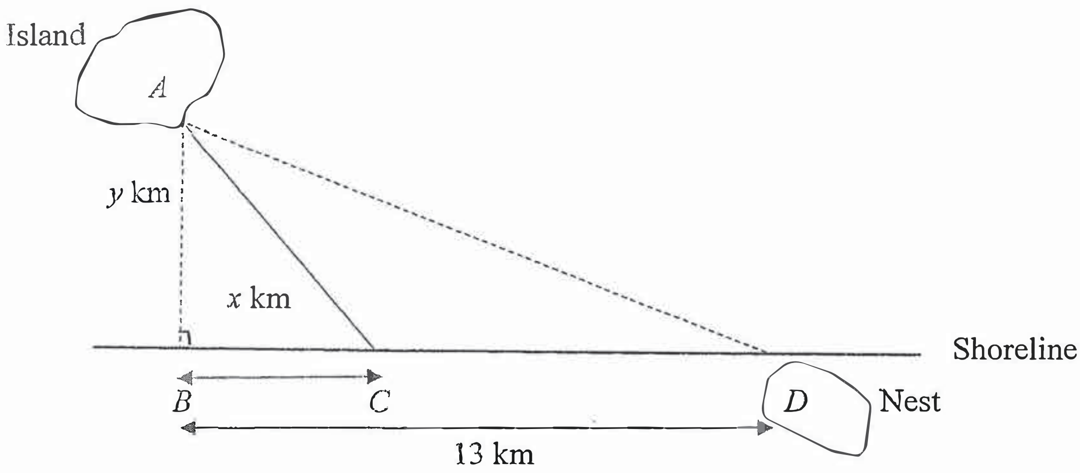
(i) Find the vector equation of the line where the two roofs meet. [3]

(ii) Show that a lamp at point P with coordinates $(2,3,-2)$ lies on the roof modelled by p_1. Find the exact perpendicular distance from P to the roof modelled by p_2. [4]

(iii) Find the acute angle formed where the two sloping roofs meet. Hence find the exact perpendicular distance from P to the line of intersection between the two roofs. [4]

(iv) A cable L passes through point $(-2,0,-2)$ and goes in the direction $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$. Assume that the width of L can be neglected. Find the cartesian equation of the plane modelling a supporting structure that contains cable L and the lamp at P. [4]

10



Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island at A that is y km from the nearest point B on a straight shoreline, flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D. Assume that the bird instinctively chooses a path that will minimise its energy expenditure. Points B and D are 13 km apart.

(a) Assume that the bird flying over land uses energy at a constant rate of k Joules/km and the energy rate of the bird flying over water is 1.4 times the energy rate flying over land. Points B and C are x km apart.

(i) If T denotes the total energy, in Joules (J), required by the bird to travel from A to D via C, show that

$$T = k[1.4\sqrt{y^2+x^2}+(13-x)].$$ [2]

[Total Energy = Energy rate × Distance]

(ii) Hence, find the value of x, in terms of y, in order to minimise the total energy required for the bird to reach D. [5]

(iii) Using $k=1$ and $y=5$ for parts **(i)** and **(ii)**, find the minimum total energy required. Sketch the graph showing the total energy (in Joules) required for the bird to reach its nest at D as the distance x varies. [4]

(b) On a separate occasion, the bird flies directly across the water to point B, then it flies towards the nest in a straight line at a constant speed of 16 km/h. Its position at any instant along BD is denoted by point P. Given that y = 5, find the rate of change of angle BAP at the instant when BP = 11 km. [5]

End of Paper

H2

TAMPINES JUNIOR COLLEGE
JC2 SEMESTRAL ASSESSMENT

MATHEMATICS — 9758/02

Paper 2 — Tuesday, 3 July 2018

3 hours

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Section A: Pure Mathematics [40 marks]

1 **(a)** The curve $y = \mathrm{f}(x)$ has a maximum point at (a, b). State the coordinates of the maximum point on the curve $y = 3\mathrm{f}(2x+1)$, in terms of a and b. [1]

(b) State a sequence of transformations that will transform the curve with equation $y = x^2$ on to the curve with equation $y = \frac{1}{4}(x-6)^2$. [2]

(c) A curve has equation $y = \mathrm{g}(x)$ where

$$\mathrm{g}(x) = \begin{cases} 4 & \text{for } 0 \le x \le 2, \\ \frac{1}{4}(x-6)^2 & \text{for } 2 < x \le 6, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch the curve with equation $y = 1 + \mathrm{g}(x)$ for $-3 \le x \le 7$. [4]

2 **(i)** Using standard series from the List of Formulae (MF26), find the first three non-zero terms in the Maclaurin series for $\mathrm{e}^x(1+\sin 2x)$, simplifying the coefficients. [3]

(ii) The three terms found in part **(i)** are equal to the first three terms in the series expansion of $(1+ax)^n$ for small x. Find the exact values of the constants a and n and use these values to find the coefficient of x^3 in the expansion of $(1+ax)^n$, giving your answer as a simplified rational number. [5]

3 Referred to the origin O, points A, B and C have position vectors **a**, **b** and $\frac{5}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$ respectively. The point P on AB is such that $AP : PB = \lambda : 1-\lambda$ and the point P on OC is such that $OP : PC = \mu : 1-\mu$.

(i) Express $\overrightarrow{OP}$ in terms of λ, **a** and **b**. [1]

(ii) By expressing $\overrightarrow{OP}$ in terms of μ, **a** and **b**, find the values of λ and μ. Hence show that P is the midpoint of OC. [4]

(iii) It is given that the position vectors of the points A and B are $2\mathbf{j} - \mathbf{k}$ and $-6\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}$ respectively. The point Q lies on OA such that PQ is perpendicular to OA. Find the position vector of the point Q. [6]

4 In a chemical reaction, the mass, x grams, of a certain salt present at time t minutes satisfies the differential equation

$$\frac{dx}{dt} = k(2 + x - x^2),$$

where $0 \le x \le 1$ and k is a constant. Initially, the mass of salt present is 1 gram and $\frac{dx}{dt} = -\frac{1}{5}$.

(i) Show that $k = -\frac{1}{10}$. [1]

(ii) By first expressing $2 + x - x^2$ in completed square form, find t in terms of x. [6]

(iii) Find the time taken for there to be no salt present in the chemical reaction. [1]

(iv) Express the solution of the differential equation in the form $x = f(t)$. Find the mass of salt present in the chemical reaction after 3 minutes. [4]

(v) Sketch the part of the curve with equation $x = f(t)$ which is relevant in this context. [2]

Section B: Probability and Statistics [60 marks]

5 A store buys 35% of its components from supplier A and the remaining 65% from supplier B. The probability that a randomly chosen component supplied by A is faulty is 0.03. The probability that a randomly chosen component supplied by B is faulty is 0.02.

(i) Find the probability that a randomly chosen component is faulty. [2]

(ii) Find the probability that a randomly chosen component was supplied by B given that it is faulty. [3]

6 Four objects a, b, c, d are to be placed into four containers A, B, C and D, with one in each container. An object is said to be correctly placed if it is put into the container of the same letter (for example: a into A), but incorrect placings otherwise. Let X be the number of correct placings and Y be the number of incorrect placings when the four objects are placed at random into the four containers.

(i) Show that $P(X = 2) = \frac{1}{4}$, and find the rest of the probability distribution of X. [5]

x	0	1	2	4
$P(X=x)$				

(ii) Find $E(X)$ and show that $Var(X) = 1$. [3]

(iii) Deduce from the results of part **(ii)** the values of $E(Y)$ and $Var(Y)$. [2]

7 For events A and B, it is given that $\mathrm{P}(A) = \frac{3}{4}$, $\mathrm{P}(A' \mid B) = \frac{1}{2}$ and $\mathrm{P}(A \cup B) = \frac{19}{20}$. Find

(i) $\mathrm{P}(A \cap B)$, [3]

(ii) $\mathrm{P}(A' \cup B')$, [2]

(iii) $\mathrm{P}(B \mid A')$. [2]

For a third event C, it is given that $\mathrm{P}(C) = \frac{1}{2}$ and that A and C are independent.

(iv) Find $\mathrm{P}(A' \cap C)$. [2]

(v) Hence state an inequality satisfied by $\mathrm{P}(A' \cap B \cap C)$. [1]

8 On average, 3.5% of a certain brand of chocolate turn out misshapen. The chocolates are sold in packets of 25.

(i) State, in context, two assumptions needed for the number of misshapen chocolates in a packet to be well modelled by a binomial distribution. [2]

(ii) Explain why one of the assumptions stated in part **(i)** may not hold in this context. [1]

Assume now that the number of misshapen chocolates in a packet has a binomial distribution.

(iii) Find the probability that a packet of 25 chocolates contains at least 1 misshapen chocolate. [1]

(iv) Ten packets of chocolate are randomly chosen. Find the probability that each packet contains at least 1 misshapen chocolate. [1]

The chocolates are packed into boxes. Each box contains 250 chocolates.

(v) Find the probability that there are at least 10 misshapen chocolates in a randomly selected box. [1]

(vi) Explain why the answer to part **(v)** is greater than the answer to part **(iv)**. [1]

(vii) The boxes are packed into cartons for shipping. However, the manufacturer is concerned that there may not be enough chocolates that are not misshapen to meet demand. The manufacturer decides that there must be a probability of at least 99.9% that a carton holds at least one box with at most 9 misshapen chocolates. What is the minimum number of boxes that the manufacturer should pack in a carton? [3]

10 A company manufactures a type of inspirational poster. The posters are rectangular in shape. The width of a poster follows a normal distribution with mean 212 mm and standard deviation 0.45 mm. The length of the poster follows an independent normal distribution with mean 298 mm and standard deviation 0.95 mm.

(i) Find the maximum length that is exceeded by at least 5% of the posters, giving your answer correct to the nearest mm. [2]

(ii) Find the probability that the total width of 4 randomly chosen posters is greater than twice the length of a randomly chosen poster by more than 250 mm. [3]

Rectangular transparent covers are manufactured to laminate the posters. The width of a cover follows a normal distribution with mean 214 mm and standard deviation 0.6 mm. The length of the cover follows a normal distribution with mean 300 mm and standard deviation 0.85 mm. The widths of a poster and a cover have independent normal distributions. The lengths of a poster and a cover have independent normal distributions. A poster and a cover are a good fit if

- the width of a cover exceeds the width of a poster by at most 2.8 mm, and
- the length of a cover exceeds the length of a poster by at most 2.8 mm.

(iii) Find the probability that a randomly chosen poster and a randomly chosen cover are a good fit. State clearly the values of the parameters of any normal distribution you use. [5]

To beautify the poster for display, the company manufactures artistically designed rectangular sheets of negligible thickness for posters to be pasted on them. The widths of the sheet, measured in mm, have the distribution $N(\mu, \sigma^2)$. It is found that 15% of the sheets have widths greater than 215.5 mm and 25% of the sheets have widths less than 213.5 mm.

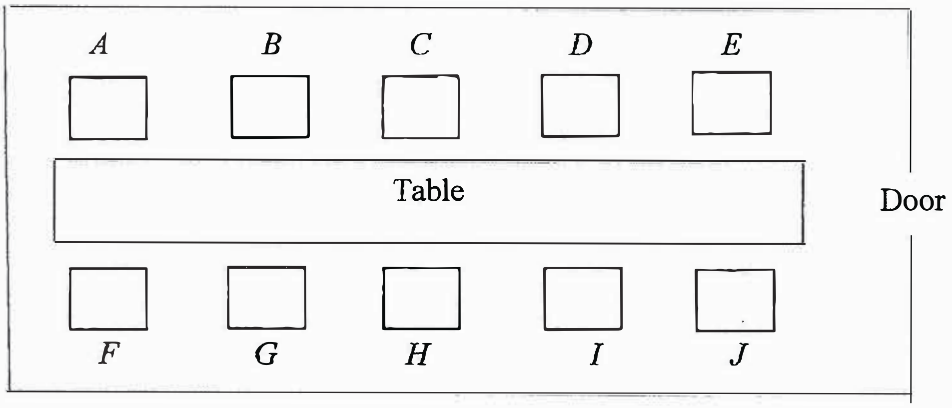
(iv) Find the values of μ and σ, correct to 2 decimal places. [3]

End of Paper

9 **(i)** A group of 6 boys and 5 girls are to be photographed together. The girls are to sit on 5 chairs placed in a row and the boys are to stand in a line behind them. Find the number of different possible arrangements. [1]

(ii) For a second photograph, the boys and girls are to be arranged with 3 boys and 3 girls standing whilst 3 boys and 2 girls are seated on the chairs in front of them, and in each row the boys and girls are to take alternate positions. Find the number of different possible arrangements. [3]

(iii) Charles, the Chairperson, is unable to attend a meeting. The other 5 boys and 5 girls proceed to the meeting room. There is a rectangular table, with a door at one end, and seats marked $A, B, C, \ldots, J$, with five seats on each side (see diagram).



In how many ways can the 5 boys and 5 girls sit

(a) if 2 particular girls must not be seated at A, E, F and J, [2]

(b) if 2 particular girls must not be seated next to each other on the same side of the table. [4]

(iv) After the meeting, the 5 boys and 5 girls proceed for lunch. Find the number of possible arrangements if the boys and girls are to take alternate seats at a round table numbered from 1 to 10. [2]