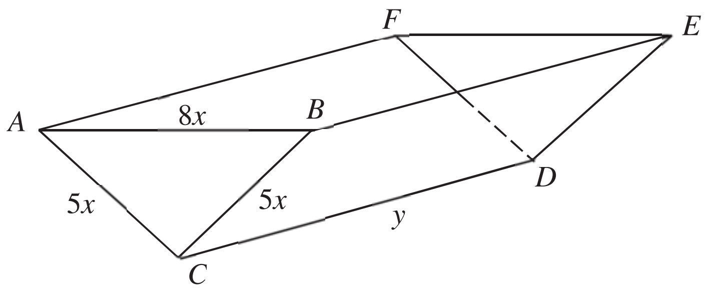
1 The curve $y=(k-6)x^2-5x$ has a minimum point. Find algebraically the set of values of k for which the curve intersects the line $y=3x-k$ at two distinct points for all real values of x. [4]

2 The curve C_1 has equation $y=\ln(x+2)$ and the curve C_2 has equation $y=\dfrac{x+2}{x-1}$.

(i) On the same diagram, sketch C_1 and C_2, stating the exact coordinates of any points of intersection with the axes and the equations of any asymptotes. [4]

(ii) Find the area of the finite region bounded by C_1, C_2 and the line $x=2$. [2]

3

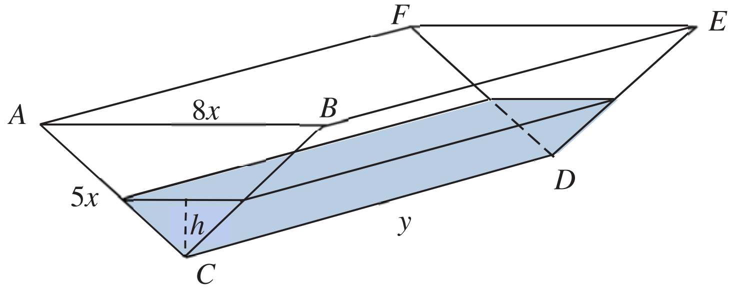


The diagram shows an open container with a rectangular top *ABEF*, constructed from 4 sheets of metal of negligible thickness. The ends *ABC* and *FED* are isosceles triangles with sides $AC = BC = ED = FD = 5x$ cm and $AB = FE = 8x$ cm. The sides *BEDC* and *AFDC* are rectangles with width $5x$ cm and length y cm. The total area of the metal sheets is 500 cm^2.

(a) **(i)** Show that the volume, V cm^3, of the container is given by $V=600x-\dfrac{144}{5}x^3$. [3]

(ii) Use differentiation to find the maximum value of V as x varies, justifying that this value is a maximum. [4]

(b)



It is given that $x = 2$.

Water is poured into the empty container at a rate of 10 cm^3/s. The water level reaches a depth of h cm after t seconds (see diagram above). Find the rate of increase of the depth of water when the volume of water in the container is 242.4 cm^3. [4]

4 **(a)** Express $\dfrac{5x+1}{(x+1)(2x+1)}$ in the form $\dfrac{A}{x+1}-\dfrac{B}{2x+1}$, where A and B are integers to be determined. Hence, differentiate $\dfrac{5x+1}{(x+1)(2x+1)}$ with respect to x. [5]

(b) Find $\displaystyle\int\left(3x+\frac{2}{x^2}\right)^2 \mathrm{d}x$. [3]

5 A stall in a food bazaar sells three types of cupcakes: vanilla, red velvet and white chocolate. The price of a vanilla cupcake, red velvet cupcake and white chocolate cupcake is \$1.50, \$3.50 and \$3.00 respectively.

On Monday, 400 cupcakes were baked. At the end of the day, $\dfrac{5}{9}$ of the vanilla cupcakes baked were sold and $\dfrac{3}{4}$ of the red velvet cupcakes baked were sold. The number of vanilla cupcakes sold was 70 more than the number of white chocolate cupcakes sold. The amount collected from selling these cupcakes was \$450.

If all the cupcakes were sold at the end of Monday, the stall would have collected \$970.

(i) By writing down three linear equations, find the number of each type of cupcake baked on Monday. [4]

(ii) Given that the production cost of a vanilla cupcake, red velvet cupcake and white chocolate cupcake is \$0.60, \$ 2.00 and \$1.80 respectively, find the profit earned by the stall on Monday and interpret the numerical value obtained in the context of the question. [2]

In order to attract more customers, the stall is trialling a new product – cookies. The cookies are sold by weight. It is predicted that the total profit \$$P$ will be related to the weight of cookies produced (x kg) by the equation

$$P=-10x^2+140x-400.$$

You may assume that all the cookies produced are sold.

(iii) Sketch the graph of P against x, stating the coordinates of the intersections with the axes. [2]

(iv) State the weight of cookies produced when the profit is a maximum. Give this value of P. [2]

(v) Give an interpretation, in context, of the value of P when $x=0$. [1]

[Turn over

Section B: Statistics (60 marks)

6 Events A and B are such that $\mathrm{P}(B)=\frac{7}{18}$, $\mathrm{P}(A \mid B)=\frac{4}{7}$ and $\mathrm{P}(A' \cap B')=\frac{1}{3}$. Find

(i) $\mathrm{P}(A \cap B)$, [1]

(ii) $\mathrm{P}(A)$. [3]

7 A username for an online portal consists of six characters. It is stipulated that the username can only contain characters chosen from the twenty-six letters of the alphabet A - Z and ten digits 0 - 9.

Kaykay is creating a username on this portal. She decides that the first four characters of her username will consist of only letters and the last two characters of the username will consist of only digits.

Suppose that repetitions are allowed, find the probability that she forms a username containing

(i) the letter K exactly once and the digits are different, [3]

(ii) the letter K as its first character or 3 as its fifth character, but not both. [3]

8 On average $100p\%$ of a certain company's pea seeds germinate. The pea seeds are sold in trays of 24.

(i) State, in context, two assumptions needed for the number of pea seeds that germinated in a tray to be well modelled by a binomial distribution. [2]

Assume now that the number of pea seeds that germinated in a tray has a binomial distribution.

(ii) The probability that 15 or 16 pea seeds germinate in a tray is 0.086550 correct to 6 decimal places. Find the value of p to a suitable degree of accuracy, given that $p > 0.5$. [3]

The trays are packed into cartons. Each carton contains 8 trays.

(iii) Find the probability that each tray in one randomly selected carton contains at least twenty pea seeds that germinated. [3]

(iv) Find the probability that there are at least 160 pea seeds that germinated in a randomly selected carton. [2]

(v) Explain why the answer to part **(iv)** is greater than the answer to part **(iii)**. [1]

9 A shop at a beach sells ice cream. For each of the nine days in July, the hours of sunshine, x, and the number of ice cream cones sold, y, are given in the table below.

x	4.3	6.9	0.0	10.4	5.2	1.8	8.0	9.2	2.1
y	74	78	13	156	80	44	134	130	55

(i) Give a sketch of the scatter diagram for the data. [2]

(ii) Find the product moment correlation coefficient. [1]

(iii) Find the equation of the regression line of y on x in the form $y = mx + c$, giving the values of m and c correct to 3 significant figures. Sketch this line on your scatter diagram. [2]

(iv) The shop closed early on one of the days as the owner wanted to attend a birthday party. Suggest, giving a reason, which day this was. [2]

(v) The shop sells 190 ice cream cones on a particular day with 9.5 hours of sunshine.

Use the equation of your regression line in part **(iii)** to calculate an estimate of the number of ice cream cones sold in a day with 9.5 hours of sunshine.

Give a reason why this estimated number is much lower than the actual number of ice cream cones sold. [2]

(vi) Use the appropriate regression line to estimate the number of hours of sunshine during a day when 100 ice cream cones are sold. Comment on the reliability of your estimate. [3]

10 A factory produces packets of a particular brand of coffee powder. Each packet is supposed to contain a mass of 800 g of coffee powder. The supervisor suspects that the machine packing the coffee powder is not operating properly. A random sample of 50 packets is taken and the masses of coffee powder in the packets are measured. The masses, x g, are summarised by

$$\sum(x-800) = -75.6, \qquad \sum(x-800)^2 = 1020.2.$$

(i) Find the unbiased estimates of the population mean and variance. [3]

(ii) Test, at the 5% significance level, whether the supervisor's suspicion is valid. [4]

(iii) State, giving a reason, whether it is necessary to assume a normal distribution for this test to be valid. [1]

After alterations to the machine, the supervisor observes that the machine seems to deliver more than 800 g of coffee in a packet. A random sample of 20 packets is selected. The sample mean is m g. A test at the 5% significance level is carried out on this sample, and the supervisor's observation is justified. Assuming that the mass of coffee powder in a packet produced after alterations to the machine is normally distributed with standard deviation 18 g, find the set of possible values of m. [5]

[Turn over

11 In a high school, the times taken, in minutes, for boys and girls to complete their 2.4 km test run have independent normal distributions with means and standard deviations as shown in the following table.

	Mean	Standard deviation
Boys	11.51	0.72
Girls	13.17	0.99

(i) Find the probability that a randomly chosen boy takes less than 10 minutes to complete the run. [1]

A boy who takes less than 10 minutes to complete the run is considered a fast runner.

(ii) A boy is chosen at random. Given that he is a fast runner, find the probability that he takes less than 9.50 minutes to complete the run. [3]

(iii) Less than 40% of the boys take more than t minutes to complete the run. Find the set of values of t. [2]

Two boys and two girls are randomly chosen. Find the probability that

(iv) the total time taken by the two boys differs from the total time taken by the two girls by at least 1 minute, [4]

(v) the average time taken by the two boys and two girls is less than the mean time taken by the boys in the high school. [4]

End of paper

1)

Minimum turning point: $k-6>0$

$k>6$

$(k-6)x^2-5x=3x-k$

$(k-6)x^2-8x+k=0$

Discriminant > 0: $b^2-4ac>0$

$(-8)^2-4(k-6)k>0$

$64-4k^2+24k>0$

$k^2-6k-16<0$

$(k-8)(k+2)<0$

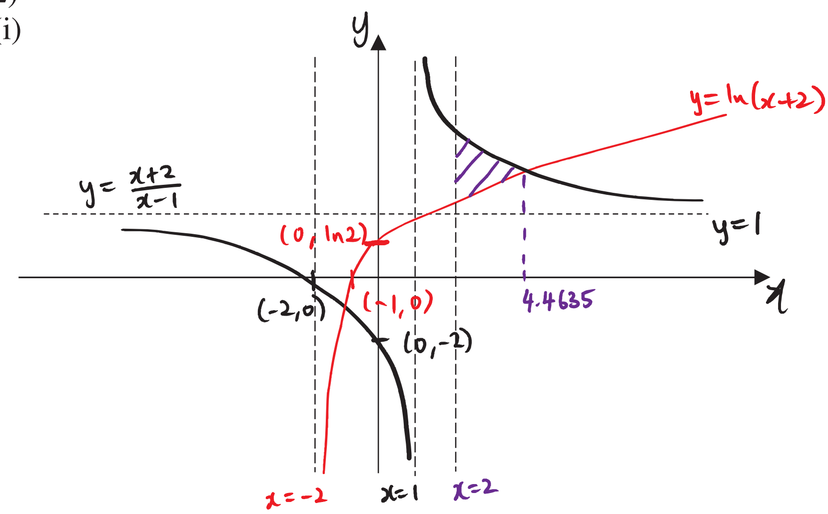
$-2<k<8$

Since $k>6$ and $-2<k<8 \Rightarrow 6<k<8$

$\{k \in \mathbb{R} : 6<k<8\}$

2)

(i)



(ii) From the GC, the point of intersection is (4.4635, 1.8662).

By GC, Area of the finite region $= \displaystyle\int_2^{4.4635} \left[\frac{x+2}{x-1} - \ln(x+2)\right] dx = 2.14$

3)

(i) Height of ABC $= \sqrt{(5x)^2-(4x)^2} = 3x$

$5xy \times 2 + \frac{1}{2}(8x)(3x) \times 2 = 500$

$10xy + 24x^2 = 500$

$y = \dfrac{500-24x^2}{10x}$

$$V=\frac{1}{2}(8x)(3x)\,y$$
$$=12x^2\left(\frac{500-24x^2}{10x}\right)$$
$$=\frac{6000x^2-288x^4}{10x}$$
$$=600x-\frac{144}{5}x^3$$

(ii)

$$\frac{dV}{dx}=600-86.4x^2$$
$$x^2=\frac{600}{86.4}=\frac{125}{18}$$

Since $x>0$, $x=2.6352$

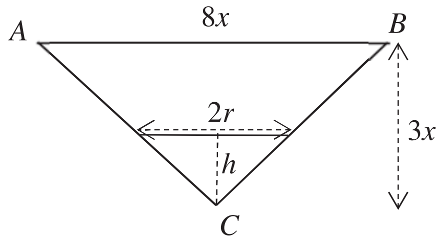
	2.6352^-	0	2.6352^+
Sign of $\frac{dV}{dx}$	+	0	-
Slope of Tangent	/	-	\

By sign test, V is maximum when $x=2.6352$.
When $x=2.6352$, $V=1054.09\approx1050$

(b) When $x=2$, $y=\frac{500-24(2)^2}{10(2)}=20.2$

$$V_w=\frac{1}{2}\times2r\times h\times20.2=20.2rh$$

$3x=6$, $8x=16$



By similar triangles:

$$\frac{h}{6}=\frac{2r}{16}$$
$$r=\frac{8}{6}h$$
$$V_w=20.2\left(\frac{8}{6}h\right)h=\frac{404}{15}h^2$$

When $V_w=242.4$, $\frac{404}{15}h^2=242.4\Rightarrow h=3$

$$\frac{dV_w}{dh}=\frac{808}{15}h\ ,\quad \frac{dV_w}{dt}=10$$
$$\frac{dh}{dt}=\frac{dh}{dV_w}\times\frac{dV_w}{dt}$$
$$=\frac{15}{808(3)}\times10=0.0619$$

4)

(a)

$$\frac{5x+1}{(x+1)(2x+1)} = = \frac{A}{x+1} - \frac{B}{2x+1}$$

$$= \frac{A(2x+1) - B(x+1)}{(x+1)(2x-1)}$$

$$= \frac{(2A-B)x + A - B}{(x+1)(2x-1)}$$

$2A - B = 5$

$A - B = 1$

Solving simultaneously, $A = 4, B = 3$

$$\frac{5x+1}{(x+1)(2x+1)} = \frac{4}{x+1} - \frac{3}{2x+1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{5x+1}{(x+1)(2x+1)}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{4}{x+1} - \frac{3}{2x+1}\right)$$

$$= -\frac{4}{(x+1)^2} + \frac{6}{(2x+1)^2}$$

(b)

$$\int\left(3x + \frac{2}{x^2}\right)^2 \mathrm{d}x = \int\left(9x^2 + \frac{12}{x} + \frac{4}{x^4}\right)\mathrm{d}x$$

$$= 3x^3 + 12\ln x - \frac{4}{3x^3} + C$$

5)

(i) Let v, r and w represent the number of vanilla cupcakes, red velvet cupcakes and white chocolate cupcakes **baked** respectively.

$v + r + w = 400 - (1)$

$1.50v + 3.50r + 3.00w = 970 - (2)$

$\frac{5}{9}v(1.50) + \frac{3}{4}r(3.50) + \left(\frac{5}{9}v - 70\right)(3.00) = 450$

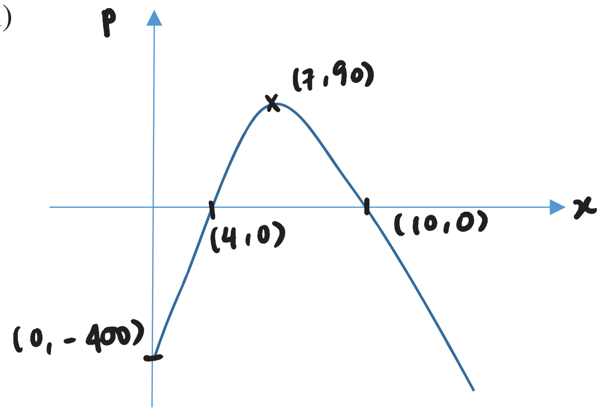
$\frac{5}{2}v + 2.625r = 660 - (3)$

By GC, $v = 180, r = 80, w = 140$

(ii) Profit $= 450 - 180(0.60) - 80(2) - 140(1.80) = -70$

The store made a loss of \$70 on Monday.

(iii)



(iv) The weight of cookies produced is 7 kg. The maximum value of P is \$90.

(v) It is the fixed cost of the stall.

6)

(i)
$$\frac{P(A \cap B)}{P(B)} = \frac{4}{7}$$
$$P(A \cap B) = \frac{4}{7}P(B)$$
$$= \frac{4}{7}\left(\frac{7}{18}\right) = \frac{2}{9}$$

(ii)
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$1 - P(A' \cap B') = P(A) + P(B) - P(A \cap B)$$
$$1 - \frac{1}{3} = P(A) + \frac{7}{18} - \frac{2}{9}$$
$$\frac{2}{3} - \frac{3}{18} = P(A)$$
$$P(A) = \frac{1}{2}$$

7)

(a) __ __ __ __ __ __

Number of usernames formed when repetitions are allowed $= 26^4 \times 10^2 = 45697600$

Number of codes with letter k exactly once and 2 different digits

$= (4 \times 1 \times 25 \times 25 \times 25) \times (10 \times 9) = 5625000$

Required probability $= \frac{5625000}{45697600} \approx 0.123$

(b) Case 1: k __ __ __ __ __

Number of usernames formed with k as its first character but not 3 as its fifth character $= 1 \times 26 \times 26 \times 26 \times 9 \times 10 = 1581840$

Case 2: __ __ __ __ 3 __

Number of usernames formed with 3 as its fifth character but not k as its first character $= 25\times26\times26\times26\times1\times10 = 4394000$

Required probability $= \dfrac{1581840+4394000}{45697600} = \dfrac{17}{130} \approx 0.131$

8)

(i)

1) The probability that a pea seed germinates is the same for all 24 seeds.
2) A pea seed germinates independently of all other pea seeds.

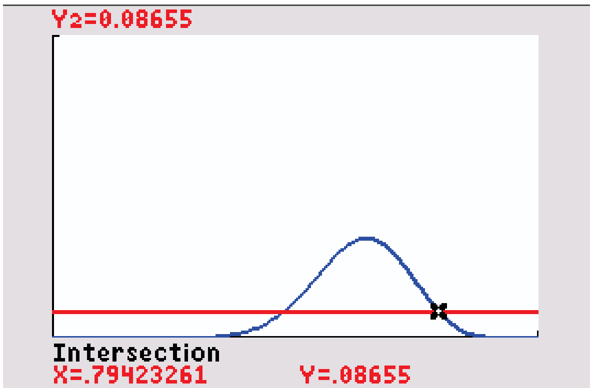
(ii)

Let X be the number of seeds that germinate out of a tray of 24.

$$X \sim \mathrm{B}(24, p)$$

$\mathrm{P}(X=15)+\mathrm{P}(X=16)=0.086550$

$\left({}^{24}C_{15}\right)(p)^{15}(1-p)^{9}+\left({}^{24}C_{16}\right)(p)^{16}(1-p)^{8}=0.086550$



By GC, $p=0.794233=0.7942$ or $p=0.476512$ (rej $\because p>0.5$)

(iii)

$\left[\mathrm{P}(X\geq20)\right]^{8}=\left[1-\mathrm{P}(X\leq19)\right]^{8}$

$=0.00121$

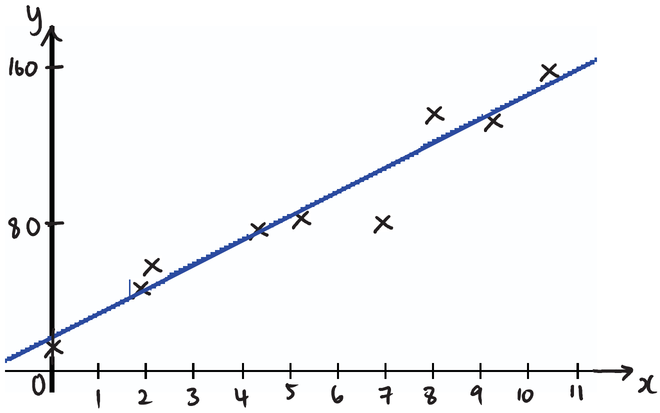
(iv) Let Y be the number of pea seeds that germinate out of 192 pea seeds in a carton.

$$Y \sim \mathrm{B}(192, 0.794233)$$

$\mathrm{P}(Y\geq160)=1-\mathrm{P}(Y\leq159)=0.103$

(v) Answer in part (iv) is greater than the answer in (iii) because the event ‘each of the 8 trays in a carton contains at least 20 pea seeds that germinated’ is a subset of the event ‘a carton contains at least 160 pea seeds that germinated’.

9)
(i), (iii)



(ii) $r = 0.965$
(iii) $y = 12.579x + 17.940$
$y = 12.6x + 17.9$

(iv) The day with 6.9 hours of sunshine. The diagram shows that point (6.9, 78) is the furthest below the regression line as compared to the other points.

(v) $y = 12.579(9.5) + 17.940 = 137$

The particular day with 9.5 hours of sunshine where 190 ice cream cones were sold might not be in July. It might be a day in December which is the holiday season and more people visit the beach. Hence, it is not appropriate to use the regression line for estimation.

OR

The owner could have ran a sale on ice cream cones.

(vi) Since *x* and *y* are measured and we need to estimate *x* given *y*, regression line of *x* on *y* is used.

$x = 0.074099y - 0.96799$

$x = 0.074099(100) - 0.96799 = 6.44$

The estimate is reliable as $r = 0.965$ is close to 1, a linear model is appropriate and $y = 100$ is within the data range of $13 \le y \le 156$.

10)
(i) Let $y = x - 800$, so

$\sum y = -75.6, \qquad \sum y^2 = 1020.2$

$$\bar{x} = \bar{y} + 800 = \frac{-75.6}{50} + 800 = 798.488$$

$$s_x^{\ 2} = s_y^{\ 2} = \frac{1}{49}\left[1020.2 - \frac{(-75.6)^2}{50}\right] = 18.488 = 18.5$$

(ii) Let μ g be the population mean mass.

$H_0 : \mu = 800$

$H_1 : \mu \neq 800$

Level of significance: 5%

Test statistics: Since $n = 50$ is sufficiently large, by Central Limit Theorem, $\overline{X}$ is approximately normal.

When H_0 is true, $Z = \dfrac{\overline{X} - 800}{S / \sqrt{50}} \sim N(0,1)$ approximately

Computation: $\overline{x} = 798.488, \quad s^2 = 18.488$

$p - \text{value} = 0.0129$

Since $p - \text{value} = 0.0129 < 0.05$, H_0 is rejected at 5% level of significance. Hence, there is sufficient evidence that the supervisor's suspicion is valid.

(iii) It is not necessary to assume the mass of a packet of coffee powder follows a normal distribution. Since sample size = 50 is sufficiently large, by Central Limit Theorem, the sample mean mass of coffee powder ($\overline{X}$) is approximately normally distributed.

$H_0 : \mu = 800$

$H_1 : \mu > 800$

Level of significance: 5%

Test statistics: When H_0 is true, $Z = \dfrac{\overline{X} - 800}{18 / \sqrt{20}} \sim N(0,1)$

Rejection region: $z \geq 1.6449$

Computation: $z = \dfrac{m - 800}{18 / \sqrt{20}}$

Owner's claim accepted

$\Rightarrow$ H_0 is rejected

$\dfrac{m - 800}{18 / \sqrt{20}} \geq 1.6449$

$m \geq 807$

$\therefore \{m \in \mathbb{Z} : m \geq 807\}$

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11)

Let X mins be the time taken by a boy and Y mins be the time taken by a girl.

$X \sim N(11.51, 0.72^2)$

$Y \sim N(13.17, 0.99^2)$

(i) $P(X < 10) = 0.017987 = 0.0180$

(ii)

$P(X < 9.5 \mid X < 10)$

$= \dfrac{P(X < 9.5)}{P(X < 10)}$

$= 0.146$

(iii)

$\mathrm{P}(X > t) < 0.4$

$1 - \mathrm{P}(X \le t) < 0.4$

$\mathrm{P}(X \le t) > 0.6$

$t > 11.7$

$\therefore \{t \in \mathbb{R} : t > 11.7\}$

(iv)

$\mathrm{E}(X_1 + X_2 - (Y_1 + Y_2)) = 2\mathrm{E}(X) - 2\mathrm{E}(Y) = -3.32$

$\mathrm{Var}(X_1 + X_2 - (Y_1 + Y_2)) = 2\mathrm{Var}(X) + 2\mathrm{Var}(Y) = 2.997$

$X_1 + X_2 - (Y_1 + Y_2) \sim \mathrm{N}(-3.32, 2.997)$

$\mathrm{P}(X_1 + X_2 \ge Y_1 + Y_2 + 1) + \mathrm{P}(X_1 + X_2 \le Y_1 + Y_2 - 1)$

$= \mathrm{P}(X_1 + X_2 - (Y_1 + Y_2) \ge 1) + \mathrm{P}(X_1 + X_2 - (Y_1 + Y_2) \le -1)$

$= 0.916$

(v)

$\bar{W} = \dfrac{X_1 + X_2 + Y_1 + Y_2}{4}$

$\mathrm{E}(\bar{W}) = \dfrac{1}{4}\left[2\mathrm{E}(X) + 2\mathrm{E}(Y)\right] = 12.34$

$\mathrm{Var}(\bar{W}) = \dfrac{1}{16}\left[2\mathrm{Var}(X) + 2\mathrm{Var}(Y)\right] = 0.1873125$

$\bar{W} \sim \mathrm{N}(12.34, 0.1873125)$

$\mathrm{P}(\bar{W} < 11.51) = 0.0276$