Topic 11 Vectors

1 The planes p_1 and p_2 have equations $\mathbf{r}.\begin{pmatrix}2\\1\\3\end{pmatrix} = 1$ and $\mathbf{r}.\begin{pmatrix}-1\\2\\1\end{pmatrix} = 2$ respectively, and meet in a line l.

(i) Find the acute angle between p_1 and p_2. [3]

(ii) Find a vector equation of l. [4]

(iii) The plane p_3 has equation $2x + y + 3z - 1 + k(-x + 2y + z - 2) = 0$. Explain why l lies in p_3 for any constant k. Hence, or otherwise, find a cartesian equation of the plane in which both l and the point (2, 3, 4) lie. [5]

(2009/P1/10)

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2 Relative to the origin O, two points A and B have position vectors given by $\mathbf{a} = 14\mathbf{i} + 14\mathbf{j} + 14\mathbf{k}$ and $\mathbf{b} = 11\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}$ respectively.

(i) The point P divides the line AB in the ratio 2 : 1. Find the coordinates of P. [2]

(ii) Show that AB and OP are perpendicular. [2]

(iii) The vector $\mathbf{c}$ is a unit vector in the direction of $\overrightarrow{OP}$. Write $\mathbf{c}$ as a column vector, and give the geometrical meaning of $|\mathbf{a}.\mathbf{c}|$. [2]

(iv) Find $\mathbf{a} \times \mathbf{p}$, where $\mathbf{p}$ is the vector $\overrightarrow{OP}$, and give the geometrical meaning of $|\mathbf{a} \times \mathbf{p}|$. Hence write down the area of triangle OAP. [4]

(2009/P2/2)

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3 The position vectors $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\mathbf{a} = 2p\mathbf{i} + 3p\mathbf{j} + 6p\mathbf{k} \quad \text{and} \quad \mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k},$$

where $p > 0$. It is given that $|\mathbf{a}| = |\mathbf{b}|$.

(i) Find the exact value of p. [2]

(ii) Show that $(\mathbf{a} + \mathbf{b}).(\mathbf{a} - \mathbf{b}) = 0$. [3]

(2010/P1/1)

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4 The line l has equation $\dfrac{x-10}{-3} = \dfrac{y+1}{6} = \dfrac{z+3}{9}$, and the plane p has equation $x - 2y - 3z = 0$.

(i) Show that l is perpendicular to p. [2]

(ii) Find the coordinates of the point of intersection of l and p. [4]

(iii) Show that the point A with coordinates $(-2, 23, 33)$ lies on l. Find the coordinates of the point B which is the mirror image of A in p. [3]

(iv) Find the area of triangle OAB, where O is the origin, giving your answer to the nearest whole number. [3]

(2010/P1/10)

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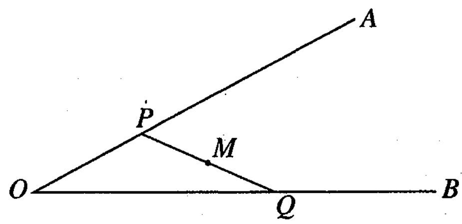

Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point P on OA is such that $OP : PA = 1 : 2$, and the point Q on OB is such that $OQ : QB = 3 : 2$. The mid-point of PQ is M (see diagram).

(i) Find $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and show that the area of triangle OMP can be written as $k|\mathbf{a} \times \mathbf{b}|$, where k is a constant to be found. [6]

(ii) The vectors $\mathbf{a}$ and $\mathbf{b}$ are now given by

$$\mathbf{a} = 2p\mathbf{i} - 6p\mathbf{j} + 3p\mathbf{k} \qquad \text{and} \qquad \mathbf{b} = \mathbf{i} + \mathbf{j} - 2\mathbf{k},$$

where p is a positive constant. Given that $\mathbf{a}$ is a unit vector,

(a) find the exact value of p, [2]

(b) give a geometrical interpretation of $|\mathbf{a}.\mathbf{b}|$, [1]

(c) evaluate $\mathbf{a} \times \mathbf{b}$. [2]

(2011/P1/7)

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6 The plane p passes through the points with coordinates $(4, -1, -3)$, $(-2, -5, 2)$ and $(4, -3, -2)$.

(i) Find a cartesian equation of p. [4]

The line l_1 has equation $\dfrac{x-1}{2} = \dfrac{y-2}{-4} = \dfrac{z+3}{1}$ and the line l_2 has equation $\dfrac{x+2}{1} = \dfrac{y-1}{5} = \dfrac{z-3}{k}$, where k is a constant. It is given that l_1 and l_2 intersect.

(ii) Find the value of k. [4]

(iii) Show that l_1 lies in p and find the coordinates of the point at which l_2 intersects p. [4]

(iv) Find the acute angle between l_2 and p. [3]

(2011/P1/11)

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7 Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ such that

$$\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k} \qquad \text{and} \qquad \mathbf{b} = \mathbf{i} + 2\mathbf{j}.$$

The point C has position vector $\mathbf{c}$ given by $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, where λ and μ are positive constants.

(i) Given that the area of triangle OAC is $\sqrt{(126)}$, find μ. [4]

(ii) Given instead that $\mu = 4$ and that $OC = 5\sqrt{3}$, find the possible coordinates of C. [4]

(2012/P1/5)

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8 **(i)** Find a vector equation of the line through the points A and B with position vectors $7\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$ and $-\mathbf{i} - 8\mathbf{j} + \mathbf{k}$ respectively. [3]

(ii) The perpendicular to this line from the point C with position vector $\mathbf{i} + 8\mathbf{j} + 3\mathbf{k}$ meets the line at the point N. Find the position vector of N and the ratio $AN : NB$. [5]

(iii) Find a cartesian equation of the line which is a reflection of the line AC in the line AB. [4]

(2012/P1/9)

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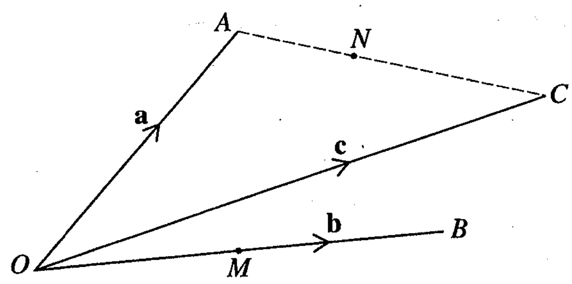

The origin O and the points A, B and C lie in the same plane, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$ (see diagram).

(i) Explain why $\mathbf{c}$ can be expressed as $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, for constants λ and μ. [1]

The point N is on AC such that $AN : NC = 3 : 4$.

(ii) Write down the position vector of N in terms of $\mathbf{a}$ and $\mathbf{c}$. [1]

(iii) It is given that the area of triangle ONC is equal to the area of triangle OMC, where M is the mid-point of OB. By finding the areas of these triangles in terms of $\mathbf{a}$ and $\mathbf{b}$, find λ in terms of μ in the case where λ and μ are both positive. [5]

(2013/P1/6)

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10 The planes p_1 and p_2 have equations $\mathbf{r}.\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 1$ and $\mathbf{r}.\begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix} = -1$ respectively, and meet in the line l.

(i) Find the acute angle between p_1 and p_2. [3]

(ii) Find a vector equation for l. [4]

(iii) The point $A\,(4, 3, c)$ is equidistant from the planes p_1 and p_2. Calculate the two possible values of c. [6]

(2013/P2/4)

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11 **(i)** Given that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, what can be deduced about the vectors $\mathbf{a}$ and $\mathbf{b}$? [2]

(ii) Find a unit vector $\mathbf{n}$ such that $\mathbf{n} \times (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = \mathbf{0}$. [2]

(iii) Find the cosine of the acute angle between $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and the z-axis. [1]

(2014/P1/3)

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12 Planes p and q are perpendicular. Plane p has equation $x + 2y - 3z = 12$. Plane q contains the line l with equation $\dfrac{x-1}{2} = \dfrac{y+1}{-1} = \dfrac{z-3}{4}$. The point A on l has coordinates $(1, -1, 3)$.

(i) Find a cartesian equation of q. [4]

(ii) Find a vector equation of the line m where p and q meet. [4]

(iii) B is a general point on m. Find an expression for the square of the distance AB. Hence, or otherwise, find the coordinates of the point on m which is nearest to A. [5]

(2014/P1/9)

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13 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C lies on OA, between O and A, such that $OC : CA = 3 : 2$. Point D lies on OB, between O and B, such that $OD : DB = 5 : 6$.

(i) Find the position vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

(ii) Show that the vector equation of the line BC can be written as $\mathbf{r} = \frac{3}{5}\lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$, where λ is a parameter. Find in a similar form the vector equation of the line AD in terms of a parameter μ. [3]

(iii) Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point E where the lines BC and AD meet and find the ratio $AE : ED$. [5]

(2015/P1/7)

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14 The line L has equation $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$.

(i) Find the acute angle between L and the x-axis. [2]

The point P has position vector $2\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$.

(ii) Find the points on L which are a distance of $\sqrt{(33)}$ from P. Hence or otherwise find the point on L which is closest to P. [5]

(iii) Find a cartesian equation of the plane that includes the line L and the point P. [3]

(2015/P2/2)

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15 The vectors $\mathbf{u}$ and $\mathbf{v}$ are given by $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = a\mathbf{i} + b\mathbf{k}$, where a and b are constants.

(i) Find $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$ in terms of a and b. [2]

(ii) Given that the $\mathbf{i}$- and $\mathbf{k}$-components of the answer to part **(i)** are equal, express $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$ in terms of a only. Hence find, in an exact form, the possible values of a for which $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$ is a unit vector. [4]

(iii) Given instead that $(\mathbf{u} + \mathbf{v}).(\mathbf{u} - \mathbf{v}) = 0$, find the numerical value of $|\mathbf{v}|$. [2]

(2016/P1/5)

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16 The plane p has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} a \\ 4 \\ -2 \end{pmatrix}$, and the line l has equation $\mathbf{r} = \begin{pmatrix} a-1 \\ a \\ a+1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$, where a is a constant and λ, μ and t are parameters.

(i) In the case where $a = 0$,

(a) show that l is perpendicular to p and find the values of λ, μ and t which give the coordinates of the point at which l and p intersect, [5]

(b) find the cartesian equations of the planes such that the perpendicular distance from each plane to p is 12. [5]

(ii) Find the value of a such that l and p do not meet in a unique point. [3]

(2016/P1/11)

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17 **(i)** Interpret geometrically the vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and t is a parameter. [2]

(ii) Interpret geometrically the vector equation $\mathbf{r}.\mathbf{n} = d$, where $\mathbf{n}$ is a constant unit vector and d is a constant scalar, stating what d represents. [3]

(iii) Given that $\mathbf{b}.\mathbf{n} \neq 0$, solve the equations $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r}.\mathbf{n} = d$ to find $\mathbf{r}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ and d. Interpret the solution geometrically. [3]

(2017/P1/6)

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18 Electrical engineers are installing electricity cables on a building site. Points (x, y, z) are defined relative to a main switching site at $(0, 0, 0)$, where units are metres. Cables are laid in straight lines and the widths of cables can be neglected.

An existing cable C starts at the main switching site and goes in the direction $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$. A new cable is installed which passes through points $P(1, 2, -1)$ and $Q(5, 7, a)$.

(i) Find the value of a for which C and the new cable will meet. [4]

To ensure that the cables do not meet, the engineers use $a = -3$. The engineers wish to connect each of the points P and Q to a point R on C.

(ii) The engineers wish to reduce the length of cable required and believe in order to do this that angle PRQ should be 90°. Show that this is not possible. [4]

(iii) The engineers discover that the ground between P and R is difficult to drill through and now decide to make the length of PR as small as possible. Find the coordinates of R in this case and the exact minimum length. [5]

(2017/P1/10)
