Section A: Pure Mathematics [40 marks]

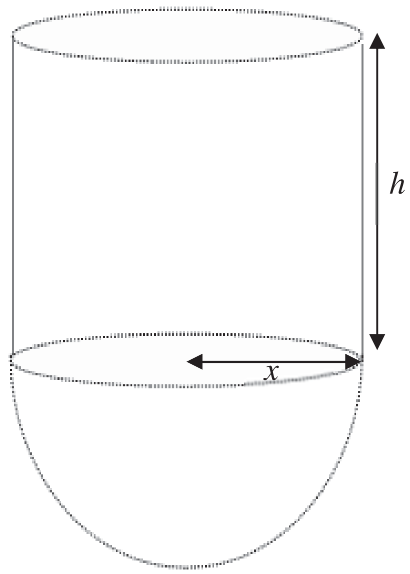
1 **(a)** **(i)** Sketch the graph of $y = 2 + \dfrac{2}{x-1}$, stating clearly the coordinates of all the points of intersection with the axes and the equations of any asymptotes. [2]

(ii) By adding a suitable graph, state the range of values of x that satisfy the inequality $\dfrac{2}{x-1} \le x$. [3]

(b) Find the set of values of p for which $x^2 - 2x + 2 = p$ has no real roots. [2]

2 [It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\dfrac{4}{3}\pi r^3$.]

A structure consists of a cylindrical top with two open ends and a hemispherical bottom, as shown in the diagram below. The cylinder has base radius x cm and height h cm. The external surface area of the structure is 40 cm^2.



(i) Show that the volume of the structure, V cm^3, is given by $V = 20x - \dfrac{1}{3}\pi x^3$. [3]

(ii) Use a non-calculator method to find the maximum value of this volume, giving your answer in the form $\dfrac{p}{q}\sqrt{\dfrac{r}{\pi}}$, where p, q and r are integers. Justify that this is the maximum value. [4]

3 **(a)** Differentiate $\dfrac{2}{e^{-x}+1}$. [2]

(b) Find $\displaystyle\int \left(1 - \frac{3}{x}\right)^2 \mathrm{d}x$. [3]

4 The curve C has equation $y = \ln(3-2x)$.

(i) Without using a calculator, find the equation of the tangent to C at the point where $x = \frac{1}{2}$, giving your answer in the form $y = mx + c$, where m and c are constants. [3]

(ii) This tangent meets the x-axis at A and the y-axis at B. Find the exact area of the triangle OAB. [3]

(iii) Find the exact area of the region bounded by C, the x-axis and the y-axis. [3]

5 One day, Kiera chanced upon an online shop which sells boys' light-weight pants. The weight of the pants varies according to the waist length of the pants as shown in the table below.

Waist length (inches)	Weight (grams)
25	170
29	200
32	232

After reading many positive online reviews about these pants, Kiera decided to buy 100 pairs of pants and try to sell them at a local flea market. As she predicted that most of the teenage boys would not be plump, she purchased four times as many smallest sized pants as the biggest sized pants. When all the pants were checked out, the total weight of the purchase was 19.12 kg and the price was \$1000.

(i) Find the number of the different sized pants that Kiera has purchased. [4]

(ii) Assuming that all the pants are sold at the same price, state the minimum price that Kiera has to sell for a pair of pants so that she would not make a loss. [1]

After the purchase of 100 pants, Kiera started on her advertising plan. She estimated that the revenue from the pants is given by $R = -0.02x^3 + 0.01x^2 + 1.2x - 1.1$, where R is the revenue in thousands of dollars and x is the amount, in hundreds of dollars, spent on advertising. Revenue is referred to as the income from the sale of goods to customers while profit is the difference between the revenue and the amount spent on buying, operating or producing something.

(iii) Sketch the graph of $R = -0.02x^3 + 0.01x^2 + 1.2x - 1.1$ for $2 \le x \le 6$. [2]

(iv) Find the minimum amount, to the nearest dollar, that Kiera needs to spend on advertising to generate a revenue of \$2150. [1]

(v) Use your calculator to estimate the maximum revenue. State the value of x for which this maximum revenue occurs. [2]

(vi) In order for Kiera to generate this maximum revenue and assuming all the pants are sold, state the price that Kiera has to sell for a pair of pants. Hence, find the profit that Kiera will make. [2]

Section B: Probability and Statistics [60 marks]

6 A school is required to send a delegation of 10 teachers to attend the Teachers' Conference 2018. This group of 10 teachers are to be selected from a pool of 7 Mathematics teachers, 5 Humanities teachers and 4 Science teachers.

(i) How many different delegations can be formed? [1]

One of the Mathematics teachers is the brother of a particular Humanities teacher.

(ii) How many different delegations can be formed such that the siblings cannot be in the same delegation? [2]

(iii) Find the probability that the delegation consists of at least 3 teachers from each department. (Mathematics, Humanities and Science are considered as 3 different departments.) [3]

7 A government agency publishes a report on smoking. In a study, 30% of smokers are classified as 'light smoker', 50% as 'moderate smoker' and 20% as 'heavy smoker'. Of those classified as 'light smoker', 'moderate smoker' and 'heavy smoker', 10%, 18% and 25% develop lung cancer in the next 10 years respectively.

(i) Draw a tree diagram to represent the above information. [2]

Find the probability that a randomly chosen smoker in the study

(ii) develops lung cancer in the next 10 years, [2]

(iii) is either a 'moderate smoker' or develops lung cancer in the next 10 years or both. [2]

A randomly chosen smoker in the study is found to have lung cancer.

(iv) Find the probability that he is a 'heavy smoker'. [2]

Two smokers in the study are randomly chosen.

(v) Find the probability that one develops lung cancer while the other does not develop lung cancer in the next 10 years. [2]

8 Hugedelay produces delay lines for use in communications. The delay times for a delay line is measured in nanoseconds (ns). It is found that 10% of the delay times are less than 274.6 ns and 7.5% are more than 288.2 ns. Assuming that the delay times for Hugedelay are normally distributed, find the mean and variance of this distribution. [4]

9 Eggs produced at a chicken farm are packaged in boxes of six. For any egg, the probability that it is broken when it reaches the retail outlet is 0.1. A box is said to be sub-standard if it contains at least two broken eggs. The number of broken eggs in a box is the random variable X.

(i) State, in context, the assumption needed for X to be well modelled by a binomial distribution. [1]

Assume now that X has a binomial distribution.

(ii) Find the probability that a randomly selected box is sub-standard. [2]

(iii) A random sample of n boxes is taken. Find the greatest value of n such that the probability that there are more than three sub-standard boxes is less than 0.01. [3]

(iv) Ten boxes are chosen at random. Find the most likely number of boxes that are sub-standard. [2]

(v) Give a possible reason in context why the assumption made in part **(i)** may not be valid. [1]

10 Gift baskets are individually wrapped to customers' requirements. The supervisor recorded the time taken by a new staff to complete wrapping his first 11 gift baskets. The data are given based on the order of wrapping the gift baskets.

Gift basket	1	2	3	4	5	6	7	8	9	10	11
No of items in gift basket, x	40	20	60	50	20	30	10	58	49	19	38
Time in min to complete wrapping, t	54.5	36.8	52.3	44.5	31.3	28.2	22.1	41.2	36.0	28.3	32.1

(i) Give a sketch of the scatter diagram for the data. Label the points from 1 to 11 according to the order of wrapping the gift basket. [2]

(ii) Find the regression line of t on x in the form of $t = ax + b$, giving the values of a and b correct to 2 decimal places. Sketch this line on your scatter diagram. Suggest a possible reason, in context, why the first five data points should be excluded. [3]

The supervisor decided to use only the last 6 data points.

(iii) Find the product moment correlation coefficient and comment on its value in the context of the data. [2]

(iv) The equation of the regression line is $t = 19.1 + 0.361x$. Estimate the time taken to wrap a gift basket with 44 items. Give two reasons why you would expect this estimate to be reliable. [2]

11 Scientists believe that the mean duration of a certain viral infection in adults is 3.6 days. From the records of a random sample of 100 infected adults, the average duration of infection is 3.42 days and the standard deviation is 1.005 days.

(i) Test, at the 5% significance level, whether the scientists' claim should be rejected. [4]

(ii) State, giving a reason, whether it is necessary to assume a normal distribution for this test to be valid. [1]

A young scientist claims that treating people with vitamin Q will reduce the average duration of infection. A second random sample of 50 adults having the infection is treated with vitamin Q. The duration of infection, y, are summarised by

$$\sum y = 166 \qquad \sum (y - \bar{y})^2 = 55$$

(iii) Find unbiased estimates of the population mean and variance using this second sample. [2]

The population mean duration of infection after being treated with vitamin Q is μ days. Using the sample data, a significance test of the null hypothesis $\mu = 3.6$ against the alternative hypothesis $\mu < 3.6$ is carried out by the young scientist at the $\alpha\%$ significance level.

(iv) Find the range of values of α for which the null hypothesis is rejected. [3]

12 Leon Fish Farm breeds only salmon and scallops. The salmon are packed in boxes of half a dozen each while the scallops are packed in boxes of 100 each. The masses, in kilograms, of salmon and scallops sold by the fish farm have independent normal distributions. The means and standard deviations of these distributions, and the selling prices, in $ per kilogram, are shown in the following table.

	Mean (kg)	Standard deviation (kg)	Selling price ($ per kg)
Salmon	5	1.5	30
Scallops	0.2	0.05	16

(i) Find the probability that a randomly chosen salmon weighs more than 5.5 kg. [1]

(ii) A customer buys 60 randomly chosen boxes of salmon. Find the probability that the average number of salmon weighing more than 5.5 kg per box is at most 2. [3]

(iii) The probability that the total mass of 2 boxes of scallops exceeding *m* kg is more than 0.95. Find the range of values of *m*. [3]

Let V be the selling price of a box of salmon.
Let W be the selling price of a box of scallops.

(iv) Find $\mathrm{P}\left(-150 < V_1 + V_2 - 6W < 150\right)$ and explain, in the context of this question, what your answer represents. [5]

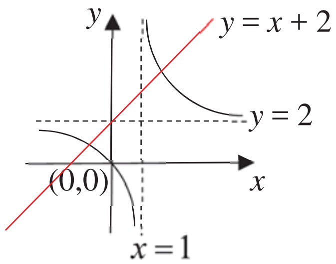
− End Of Paper −

Yishun Junior College
2018 JC2 H1 Math Preliminary Examination
Solutions

No	Solutions

1

(i) $y = 2 + \dfrac{2}{x-1}$



(ii)

$$\frac{2}{x-1} \le x$$

$$2 + \frac{2}{x-1} \le x + 2$$

Additional graph to add in: $y = x + 2$

Intersection points between the 2 graphs:

$(-1,1)$ and $(2,4)$

Hence,

$-1 \le x < 1$ or $x \ge 2$

(b)

$x^2 - 2x + 2 - p = 0$

Since there are no real roots,

Discriminant < 0

$(-2)^2 - 4(1)(2-p) < 0$

$4 - 8 + 4p < 0$

$4p < 4$

$\{p \in \mathbb{R} : p < 1\}$

2	(i) $2\pi x^2 + 2\pi x h = 40$ $h = \frac{40 - 2\pi x^2}{2\pi x}$ Volume of the structure $= \frac{1}{2} . \frac{4}{3} \pi x^3 + \pi x^2 h$ $= \frac{2}{3} \pi x^3 + \pi x^2 \left(\frac{40 - 2\pi x^2}{2\pi x} \right)$ $= \frac{2}{3} \pi x^3 + x\left(20 - \pi x^2\right)$ $= 20x - \frac{1}{3} \pi x^3$ (shown) (ii) $\frac{\mathrm{d}V}{\mathrm{d}x} = 20 - \pi x^2$ At stationary value, $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$. $20 - \pi x^2 = 0$ $\therefore x = \sqrt{\frac{20}{\pi}}$ as $x > 0$ When $x = \sqrt{\frac{20}{\pi}}^{-}$, $V = 20\sqrt{\frac{20}{\pi}} - \frac{1}{3}\pi\left(\sqrt{\frac{20}{\pi}}\right)^3$ $= \frac{40}{3}\sqrt{\frac{20}{\pi}}$ $= \frac{80}{3}\sqrt{\frac{5}{\pi}}$

x	$\sqrt{\frac{20}{\pi}}^{-}$	$\sqrt{\frac{20}{\pi}}$	$\sqrt{\frac{20}{\pi}}^{+}$
$\frac{\mathrm{d}V}{\mathrm{d}x}$	+ve	0	-ve
slope	/	-	\

Hence, volume is maximum.

3	(a) $\frac{2}{e^{-x}+1}$ $\frac{d\left[2\left(e^{-x}+1\right)^{-1}\right]}{dx}$ $= -2\left(e^{-x}+1\right)^{-2}\left(-e^{-x}\right)$ $= 2e^{-x}\left(e^{-x}+1\right)^{-2}$ (b) $\int\left(1-\frac{3}{x}\right)^2 dx$ $= \int\left(1-\frac{6}{x}+\frac{9}{x^2}\right)dx$ $= x-6\ln\lvert x\rvert-\frac{9}{x}+C$
4	(i) $y=\ln(3-2x)$ $\frac{dy}{dx}=\frac{-2}{3-2x}$ When $x=\frac{1}{2}$, $y=\ln\left[3-2\left(\frac{1}{2}\right)\right]=\ln 2$, $\frac{dy}{dx}=\frac{-2}{3-2\left(\frac{1}{2}\right)}=-1$ Equation of tangent: $y-\ln 2=-1\left(x-\frac{1}{2}\right)$ $y=-x+\frac{1}{2}+\ln 2$ (ii) When $y=0$, $x=\frac{1}{2}+\ln 2$ $A\left(\frac{1}{2}+\ln 2, 0\right)$ When $x=0$, $y=\frac{1}{2}+\ln 2$ $B\left(0, \frac{1}{2}+\ln 2\right)$

area of triangle $OAB = \frac{1}{2}\left(\frac{1}{2}+\ln 2\right)\left(\frac{1}{2}+\ln 2\right)$

$$= \frac{1}{2}\left(\frac{1}{2}+\ln 2\right)^2 \text{ units}^2$$

(iii) Required area $= \displaystyle\int_0^{\ln 3} \frac{1}{2}\left(3-e^y\right)\, dy$

$$= \frac{1}{2}\left[3y-e^y\right]_0^{\ln 3}$$

$$= \frac{1}{2}\left[3\ln 3-e^{\ln 3}-0+1\right]$$

$$= \frac{1}{2}\left[3\ln 3-2\right] \text{ units}^2$$

5 (i)

Let x, y, and z be the number of 25-inch, 29-inch and 32-inch pants respectively.

$x+y+z=100$

$x-4z=0$

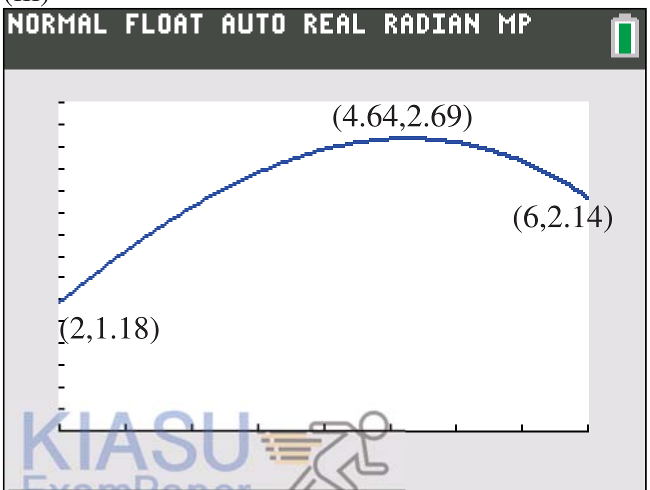
$170x+200y+232z=19120$

Solving,

$x = 40$, $y = 50$, $z = 10$

(ii) Minimum price to sell per pair = \$10

(iii)



(iv) When $R = 2.15$, $x = 3.1438579$

Minimum amount spent = \$315

(v) Maximum point (4.6419, 2.68535)

Maximum revenue = \$2685.35 when $x = 4.64$

(vi) Price per pair = $\dfrac{2685.35}{100} = 26.85$

At maximum revenue, \$464 is spent on advertising.

Profit made

$= 2685.35 - 1000 - 464.19$

= \$1221.16

6 (i) No of different delegations

$= {}^{16}C_{10} = 8008$

(ii) <u>Method 1</u>

No of different delegations

$= {}^{14}C_{9} \times 2 + {}^{14}C_{10}$

$= 5005$

<u>Method 2</u>

No of different delegations

$= {}^{16}C_{10} - {}^{14}C_{8}$

$= 5005$

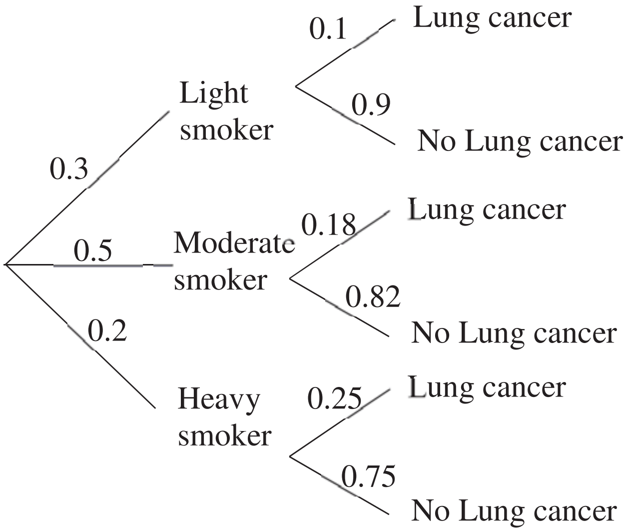
(iii) Required probability

$$= \frac{{}^{7}C_{4} \times {}^{5}C_{3} \times {}^{4}C_{3} + {}^{7}C_{3} \times {}^{5}C_{4} \times {}^{4}C_{3} + {}^{7}C_{3} \times {}^{5}C_{3} \times {}^{4}C_{4}}{{}^{16}C_{10}}$$

$$= \frac{2450}{8008} = \frac{175}{572}$$

Or 0.306 (3 s.f)

7 (i)



(ii) Required probability

$= 0.3\times0.1 + 0.5\times0.18 + 0.2\times0.25$

$= 0.17$

(iii) Required probability

$= 0.5 + 0.17 - 0.5\times0.18$

$= 0.58$

(iv)

$\text{P(Heavy smoker} \mid \text{Develops lung cancer)}$

$= \dfrac{0.2\times0.25}{0.17}$

$= \dfrac{5}{17}$ or 0.294

Required probability

$= 0.17\times(1-0.17)\times2$

$= \dfrac{1411}{5000}$ or 0.2822

8 Let X be the delay times (in ns) for a delay line.

and $X \sim \mathrm{N}(\mu, \sigma^2)$

Given that $\mathrm{P}(X < 274.6) = 0.1$

$$P\left(Z < \frac{274.6-\mu}{\sigma}\right) = 0.1$$

$$\frac{274.6-\mu}{\sigma} = -1.28155167$$

$\mu - 1.28155167\sigma = 274.6$ ------ (1)

P($X > 288.2$) = 0.075

$$P\left(Z > \frac{288.2-\mu}{\sigma}\right) = 0.075$$

$$\frac{288.2-\mu}{\sigma} = 1.439531471$$

$$\mu + 1.439531471\sigma = 288.2 \quad \text{-------} \ (2)$$

From GC, μ = 281.0051152
σ = 4.998074

i.e. Mean =281 (3 sf)
Variance = σ^2 = 25.0 (3 sf)

9

(i) Assumption: Whether or not an egg is broken is independent of another egg.
(ii) $X \sim$ B(6, 0.1)
P(a box of eggs is sub-standard) = P($X \geq 2$)
= 1 – P ($X \leq 1$)
= 0.114 (3 sf)

(iii) Let W be the number of boxes that are sub-standard out of n.
W~N(n, 0.114265)
Given P($W > 3$) < 0.01
1 – P($W \leq 3$) < 0.01

From GC,
When n = 8, P($W > 3$) = 0.00815 < 0.01
When n = 9, P($W > 3$) = 0.01336 > 0.01
Therefore, greatest value of number of boxes chosen is 8.

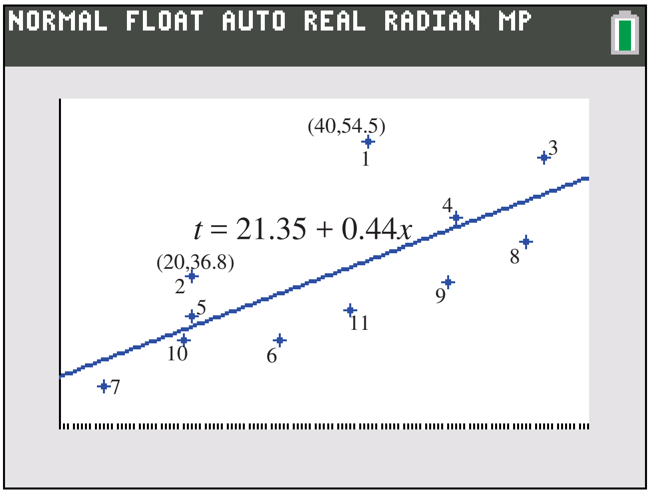
(iv) Let Y be the number of boxes that are sub-standard out of 10.
$Y \sim$ B(10, 0.114265)

P(Y = 0) = 0.2972
P(Y = 1) = 0.3834
P(Y = 2) = 0.2226
i.e. Most likely number = 1

(v) Breakages are not independent of each other. (if one egg in a box is broken, it is more likely the others will be).

10

(i) Scatter diagram



(ii) From GC the required regression line is
$t = 21.3521 + 0.43763x$
$t = 21.35 + 0.44x$ (2dp)

This is likely because the new staff was not familiar with the wrapping process initially for the first 5 gift baskets.

(iii) Considering only the last 6 points,

$r = 0.974$.

This shows that there is a strong positive linear correlation between x and t. As the number of items in the gift basket increases, the timing to complete wrapping increases.

(iv) When $x = 44$, $t = 19.1 + 0.361(44) = 34.984$.

i.e. The estimated wrapping time for 44 items is 35.0 mins.

The estimate is reliable because it is obtained by interpolation and $r = 0.974$ is close to +1.

11

Let X be the duration of a certain viral infection in an adult.
(i) H_0: $\mu = 3.6$
H_1: $\mu \neq 3.6$
Under H_0, test statistic is,

$$Z = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim N(0,1) \text{ approximately (by CLT)}$$

where μ=3.6, $s^2 = \frac{100}{99}(1.005)^2$, $\overline{x} = 3.42$, n=100
From GC, p-value = 0.0747382

Since p-value = 0.0747382 > 0.05, we do not reject H_0, and conclude that, at 5% level there is no significant evidence that the average duration of infection is different from the scientist's claim.

(ii) Not necessary. Since $n = 100$ is large, by CLT, the sample mean duration of a certain viral infection in an adult follows a normal distribution approximately.

(iii) unbiased estimate of population mean, $\bar{y} = \dfrac{\sum y}{50} = \dfrac{166}{50} = 3.32$

unbiased estimate of population variance,

$$s^2 = \frac{\sum (y - \bar{y})^2}{49} = \frac{55}{49}$$

(iv) H_0: $\mu = 3.6$

H_1: $\mu < 3.6$

Under H_0, test statistic is,

$Z = \dfrac{\bar{X} - \mu}{S/\sqrt{n}}$ ~ N(0,1) approximately (by CLT) where μ=3.6, $s^2 = \dfrac{55}{49}$, $\bar{x} = 3.32$, $n = 50$

From GC, p-value = 0.0308262

Given null hypothesis is rejected,

p-value = $0.0308262 < \dfrac{\alpha}{100}$

i.e. $\alpha > 3.08262$

Ans: $\alpha \geq 3.09$

12

Let S be the mass (in kg) of a salmon,

Let A be the mass (in kg) of a scallop.

$S \sim N(5, 1.5^2)$, $A \sim N(0.2, 0.05^2)$

(i) $P(S > 5.5) = 0.369$

(ii) Let X be the number of salmon that weigh more than 5.5 kg in a box of 6.

$X \sim B(6, 0.3694414)$

$E(X) = 6(0.3694414) = 2.2166484$

$Var(X) = 6(0.3694414)(0.6305586) = 1.3977267$

Since $n = 60$ is large, by CLT,

Let $\bar{X} = \dfrac{X_1 + X_2 + \cdots + X_{60}}{60}$

$\bar{X} \sim N\left(2.2166484, \dfrac{1.3977267}{60}\right)$ approximately

$P(\bar{X} \leq 2) = 0.0779$

	(iii) Let $G = A_1 + A_2 + \ldots + A_{200}$ $G \sim \text{N}(200(0.2), 200(0.05)^2)$ i.e. $G \sim \text{N}(40, 0.5)$ $\text{P}(G > m) > 0.95$ $1 - \text{P}(G \leq m) > 0.95$ $\text{P}(G \leq m) < 0.05$ $m < 38.8369$ $m < 38.8$ (iv) $V = 30(S_1 + S_2 + \ldots + S_6) \sim \text{N}(30(6\times5), 30^2(1.5)^2\times6)$ i.e. $V \sim \text{N}(900, 12150)$ $W = 16(A_1 + A_2 + \ldots + A_{100}) \sim \text{N}(16(100\times0.2), 16^2(0.05)^2\times100)$ i.e. $W \sim \text{N}(320, 64)$ $V_1 + V_2 - 6W \sim \text{N}(2(900) - 6\times320, 2(12150)+6^2(64))$ i.e. $V_1 + V_2 - 6W \sim \text{N}(-120, 26604)$ $\text{P}\left(-150 < V_1 + V_2 - 6W < 150\right) = 0.524$ It is the probability that the selling price of 2 boxes of salmon differs from six times the selling price of a box of scallops by less than \$150. Or It is the probability of the 'the difference between the selling price of 2 boxes of salmon and six times the selling price of a box of scallops is within ± 150.