A. Algebraic questions

1. **[MI 14 Prelims]**
   The first, third and fourth terms of an arithmetic progression are consecutive terms of a geometric progression.
   (a) Show that $a = -4d$, where a and d represent the first term and common difference of the arithmetic progression respectively. [2]
   (b) Hence determine if the geometric progression is convergent. [1]

2. **[VJC 17 Prelims]**
   A geometric series has common ratio r, and an arithmetic series has first term a and common difference d, where a and d are non-zero and $a > 0$. The first three terms of the geometric series are equal to the first, eighth and thirteenth terms respectively of the arithmetic series.
   (a) Show that $7r^2 - 12r + 5 = 0$. [2]
   (b) Deduce that the geometric series is convergent. [2]
   (c) The sum of the first n terms of the geometric series is denoted by S_n. Find the smallest value of n for S_n to be within 0.1% of the sum to infinity of the geometric series. [4]
   (d) Find exactly the sum of the first 2017 terms of the arithmetic series, leaving your answer in terms of a. [3]

3. **[IJC 13 Prelims]**
   An arithmetic progression has first term a and common difference d, where a and d are non-zero. The second, seventh and ninth terms of the arithmetic progression are consecutive terms of a geometric progression. Find, in terms of d, the sum of the first 20 odd-numbered terms of the arithmetic progression. [4]

4. **[PJC 18 Prelims]**
   The product of the first three terms of a convergent geometric progression is 1000. If 6 is added to the second term and 7 is added to the third term, the three terms are now consecutive terms of an arithmetic progression. Find the first term and common ratio of the geometric progression. [4]

B. Show questions

5. **[NJC 14 Prelims]**
   The sum of the first n terms of a series is given by $10 - \dfrac{2^{n+1}}{5^{n-1}}$. Show that the series is a geometric series and state the value of the common ratio. [4]

6. **[MI 14 Prelims]**
   (a) Given that T_n represents the nth term of an arithmetic progression, show that the terms of a new sequence where the nth term is represented by $U_n = \mathrm{e}^{T_n}$ forms a geometric progression. [2]
   (b) Given that X_n represents the nth term of a geometric progression, show that the terms of a new sequence where the nth term is represented by $Y_n = \ln X_n$ forms an arithmetic progression. [2]

7. **[MJC 18 Prelims]**
   The sum of the first n terms of a sequence $\{u_n\}$ is given by $S_n = kn^2 - 3n$, where k is a non-zero real constant.
   (a) Prove that the sequence $\{u_n\}$ is an arithmetic sequence. [3]
   (b) Given that u_2, u_3 and u_6 are consecutive terms in a geometric sequence, find the value of k. [3]

C. Problem Sums

8. **[MJC 18 Prelims]**
   A zoology student observes jaguars preying on white-tailed deer in the wild. He observes that when a jaguar spots its prey from a distance of d m away, it starts its chase. At the same time, the white-tailed deer senses danger and starts escaping. He models the predator-prey movements as follows:

   The jaguar starts its chase with a leap distance of 6 m. Subsequently, each leap covers a distance of 0.1 m less than its preceding leap.

   The white-tailed deer starts its escape with a leap distance of 9 m. Subsequently, each leap covers a distance of 5% less than its preceding leap.
   (a) Find the total distance travelled by a white-tailed deer after n leaps. Deduce the maximum distance travelled by a white-tailed deer. [3]
   (b) Assume that both predator and prey complete the same number of leaps in the same duration of time. Given that $d = 11$ m, find the least value of n for a jaguar to catch a white-tailed deer within n leaps. [3]

9. **[NJC 14 Prelims]**
   In Marathon A, there are 20 participants. The prizes are awarded according to the following rule: the first prize is \$1000, the second prize is $\frac{4}{5}$ of the first prize, the third prize is $\frac{4}{5}$ of the second prize, and so on, till the 20th prize.
   (a) What is the total amount of money that the organizers need to have for the prize fund? [2]
   (b) What is the assumption needed for your calculation in part (a) to be valid? [1]

   In Marathon B, the prizes are awarded according to the following rule: the last prize is \$15, the second last prize is \$185 more than the last prize, the third last prize is \$185 more than the second last prize, and so on, till the first prize. Determine the maximum number of prizes to be awarded if the organizers have only $\$100,000$ to sponsor the prize fund. [2]

10. **[JJC 14 Prelims]**
An athlete hopes to represent Singapore at the SEA Games in 2015 and he embarks on a rigorous training program. On his first training session, he ran a distance of 7.5 km. For his subsequent training sessions, he ran a distance of 800 m more than the previous training session.

(a) Express, in terms of n, the distance (in km) he ran on his nth training session. [2]

(b) Find the minimum number of training sessions required for him to run a total distance of at least 475 km. [3]

After a month, he realized that his progress was unsatisfactory and he decided to modify the training. For the modified training programme, he ran a distance of x km for the first session, and on each subsequent training session, the distance covered is $\frac{6}{5}$ times of the previous session.

(c) Find x, to the nearest integer, if he covered a distance of 14.93 km on the 6th training session. [2]

(d) Using the answer in (c), and denoting the total distance after n training sessions by G_n, write down an expression for G_n in terms of n.

Hence show that $\sum_{n=1}^{N} G_n$ may be expressed in the form $aG_N + bN$, where a and b are integers to be determined. [4]

11. **[RVHS 13 Prelims (adapted)]**
At the end of December 2012, John's company managed to secure a long term city development contract with the local government. He thus decided to approach Companies A and B for supply of construction raw materials.

(a) After the first supply of 200 units of raw materials at the end of January 2013, Company A supplied 20 units more than it supplied at the end of the previous month. For example, Company A supplied 220 units at the end of February 2013 and 240 units at the end of March 2013. However, John decided that he would end the partnership with Company A at end of February 2017. Determine

i. the total amount of raw materials that would be supplied by Company A by the end of February 2017, [2]

ii. the time when Company A would have first supplied more than half of the total amount supplied for the whole duration. [2]

(b) Company B also supplied Johns company with raw materials at the end of each month from January 2013. The amount supplied for each month is in a geometric progression. In particular, Company B would supply 387.24 units in December 2013 and 109.37 units in December 2014. Find, correct to the nearest unit,

i. the amount of raw materials supplied by Company B at the end of January 2013, [3]

ii. the total amount of raw materials Company B could supply in the long run. [3]

(c) Determine the date when the total amount of raw materials supplied by Company A first exceeds that of Company B. [2]

12. **[CJC 17 Prelims (modified)]**
Kumar wishes to purchase a gift priced at \$260 for his mother. He starts with \$100 in his piggy bank on 1st January 2017. Subsequently,

- Kumar donates 30% of his money in his piggy bank on the 15th of every month;
- Kumar saves \$120 in his piggy bank on the 1st of every month, starting from 1st February 2017.

(a) Find the amount of money in Kumar's piggy bank at the end of March 2017. [2]

(b) Show that the amount of money in Kumar's piggy bank at the start of the n months is $280 - 300(0.7)^n$. [3]

(c) At the end of which month will Kumar first be able to purchase the gift for his mother? [2]

13. **[TPJC 17 Prelims]**
Timber cladding is the application of timber planks over timber planks to provide the layer intended to control the infiltration of weather elements.

(a) Using method A, 20 rectangular planks are used and the lengths of the planks form an arithmetic progression with common difference d cm. The shortest plank has length 65 cm and the longest plank has length 350 cm.

i. Find the value of d. [2]

ii. Find the total length of all the planks. [2]

(b) Using method B, a long plank of 2000 cm is sawn off by a machine into n smaller rectangular planks. The length of the first plank is a cm and each successive plank is $\frac{8}{9}$ as long as the preceding plank.

i. Show that the total length of the planks sawn off can never be greater than k times the length of the first plank, where k is an integer to be determined. [2]

ii. Given that $a = 423$, find the greatest possible integral value of n and the corresponding length of the shortest plank. [4]

14. **[SRJC 17 Prelims]**

(a) The fifth, ninth and eleventh terms of a geometric progression are also the seventh, twenty-fifth and forty-ninth terms of an arithmetic progression with a non-zero common difference respectively.

Show that $3R^6 - 7R^4 + 4 = 0$ where R is the common ratio of the geometric progression and determine if the geometric progression is convergent. [4]

(b) A semicircle with radius 12 cm is cut into 8 sectors whose areas follow a geometric progression. The first sector, which is the largest, has an area of A cm^2. The second sector has an area of Ar cm^2, the third sector has an area of Ar^2 cm^2, and so on, where r is a positive constant. Given also that the total area of the odd-numbered sectors is 10π cm^2 more than that of the even-numbered sectors, find the values of A and r. [5]

(c) The production levels of a particular coal mine in any year is 4% less than in the previous year. Show that the total production of the coal mine can never exceed 25 times the production in the first year. [2]

15. **[PJC 18 Prelims]**

(a) A man takes a loan of \$P for a house from a bank at the beginning of a month. The interest rate is 0.5% per month so that at the start of every month, the amount of outstanding loan is increased by 0.5%. Equal instalment is paid to the bank at the end of every month. Find his monthly instalment if he would like to repay the loan in 20 years, leaving your answer in the form $0.005P\frac{k^n}{k^n-1}$, where k and n are constants to be determined. [5]

(b) If the man takes a loan of \$1 000 000 for the house from another bank which charges interest at the start of every month, he will have to pay a monthly instalment of \$10 000 at the end of every month over 20 years to repay the loan. Find the monthly interest rate charged by this bank. [3]

Answers

1. Convergent
2. (b) $r = \frac{5}{7}$ so $-1 < r < 1$. Hence the geometric series is convergent.
   (c) Smallest $n = 21$.
   (d) $d = -\frac{2}{49}a$, $S_n = -\frac{566777}{7}a$.
3. $a = 20, r = \frac{1}{2}$.
4. $\frac{1700}{3}d$.
5. $\frac{2}{5}$.
7. $k = \frac{3}{2}$.
8. (a) $S_n = 180(1 - 0.95^n)$.
   (b) Minimum $n = 50$.
9. \$4942.35.
   33 prizes.
10. (a) $6.7 + 0.8n$.
    (b) Least $n = 27$.
    (c) $x = 6$.
    (d) $30\left((\frac{6}{5})^n - 1\right)$.
    $6G_N - 30N$.
11. (a) i. 34 000.
    ii. End of October 2015 ($n = 34$).
    (b) i. 1 234.
    ii. 12 340.
    (c) End of February 2015 ($n = 26$).
12. (a) \$177.10.
    (c) August 2017.
13. (a) i. $d = 15$.
    ii. $S_{20} = 4150$.
    (b) i. $k = 9$.
    ii. $n = 6$, length = 235 cm.
14. (a) $|R| = 2 > 1$. Hence not convergent.
    (b) $r = 0.75610, A = 61.8$.
15. (a) $k = 1.005, n = 240$.
    (b) 0.877% per month.