Questions

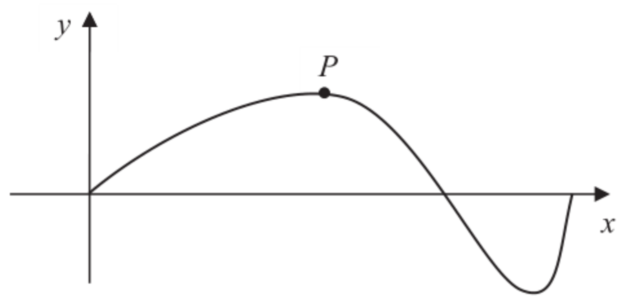
1.

(i) Find $\int \sin\theta\cos^2\theta \, \mathrm{d}\theta$. [2]

(ii) The diagram below shows the curve defined by the parametric equations

$$x = \sin\theta + \theta, \; y = \sin\theta\cos\theta,$$

for $0 \le \theta \le \pi$.



The curve has a maximum point at P.

Find the exact area of the region bounded by the curve, the tangent at P and the line $x = \pi$. [5]

9.

(a) **(i)** Find the values of a, b and c such that $x = (ax+b)(x+1) + c(x^2+2x+2)$ for all real values of x. [2]

(ii) Hence find $\int \frac{x}{(x+1)(x^2+2x+2)} \, \mathrm{d}x$. [4]

(b) **(i)** By using a graphical approach, solve $2x + \sqrt{x+5} > 0$. [1]

(ii) Given that $\int_{-2}^{2} \left|2x + \sqrt{x+5}\right| \mathrm{d}x = p\sqrt{3} + \frac{1}{3}q\left(\sqrt{7}-1\right)$ for some real values of p and q, find the values of p and q. [3]

Integration, volume, DE selected questions

2.

(a) Find $\int \sin 2x \sin 3x \, dx$. [3]

(b) **(i)** Find $\int (\ln x)^2 \, dx$. Hence find the exact value of $\int_1^e (\ln x)^2 dx$. [4]

The equation of a curve C is given by $y = e^x$. The line l with the equation $y = ex$ is tangential to C at the point $(1, e)$. The region R is bounded by the curve C, the line l and the y-axis.

(ii) Find the exact value of the volume of revolution when R is rotated 2π radians about the y-axis. [3]

3.

(a) **(i)** Show that $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ using the addition formula from the List of Formulae (MF26).

[1]

(ii) By using the substitution $x = \tan\theta$, or otherwise, find the exact value of $\int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$.

[5]

(b) Find the exact value of $\int_{-1}^{1} \frac{2+|x|}{2+x} dx$. [4]

4. Scientists are researching on the spread of a communicable disease in a particular town. The town has a population of 5,750 people. A scientist proposes that the rate of growth of the infected population, x, at time t days after the initial outbreak, is proportional to the product of the infected population and the non-infected population. It is noted that there are 750 infected people at the initial outbreak and it is increasing at a rate of 500 people per day.

By setting up and solving a differential equation, show that $x = \frac{17250}{3+20e^{-\frac{p}{q}t}}$, where p and q are positive integers to be determined. [8]

5.

On Island B, he found that the annual natural growth rate of the population of the squirrel is 50. However, due to the industrial activities taking place on the island, the squirrels are dying at a rate proportional to number of squirrels present at time t.

(i) It is known that the population will remain constant when there are 5000 squirrels. Find the differential equation relating $\frac{\mathrm{d}n}{\mathrm{d}t}$ and n. [3]

(ii) It is given that there are 2000 squirrels initially on Island B. Find n in terms of t. [3]

(iii) Explain what will eventually happen to the population of squirrels on Island B. [1]

6.

A zoologist has been studying the change in the population of a certain species of squirrel of size n at time t years in each of the 2 islands A and B.

(a) In Island A, he found that the population of the squirrel can be modelled by the differential equation

$$e^{n} + te^{n}\frac{\mathrm{d}n}{\mathrm{d}t} = 4t, \quad \text{where } t \geq 1 .$$

(i) Using the substitution $y = 2te^{n}$, show that the differential equation can be reduced to $\frac{\mathrm{d}y}{\mathrm{d}t} = kt$, where k is a positive constant to be determined. [2]

(ii) It is given that when $t = 1$, there are no squirrels on Island A. Sketch the solution curve that shows how the population of squirrels on Island A changes with respect to time t in the context of the question. [4]

Answers

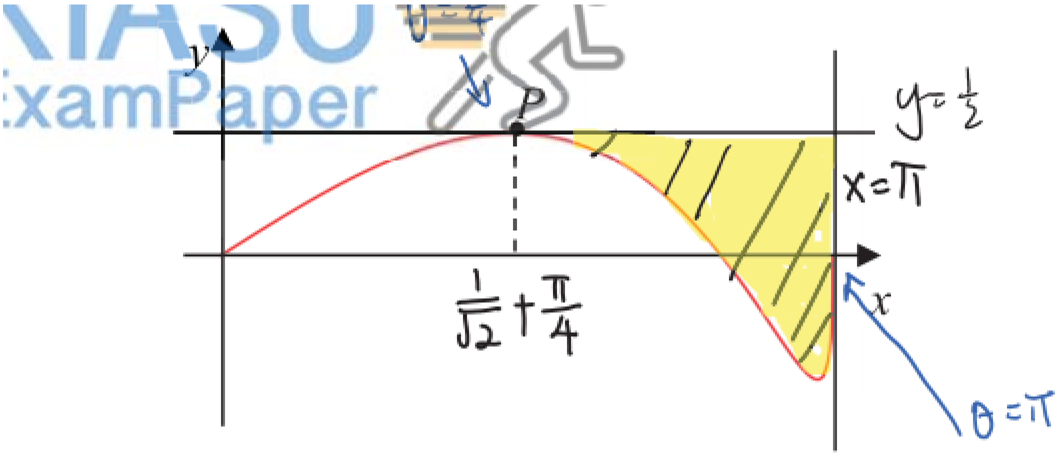
1. (i)

$$\int \sin\theta\ (\cos\theta)^2 \mathrm{d}\theta, \qquad \left[\text{use } \int f'(\theta)[f(\theta)]^n \mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1} + c\right]$$

$$= -\int (-\sin\theta)(\cos\theta)^2 \mathrm{d}x, \qquad \text{let } f(\theta) = \cos\theta,\ f'(\theta) = -\sin\theta$$

$$= -\frac{1}{3}(\cos\theta)^3 + C, \qquad \text{where } C \text{ is an arbitrary constant.}$$

1. (ii)



$$\theta = \frac{\pi}{4}$$

$$\text{Area of required region} = \int_{x=\frac{1}{\sqrt{2}}+\frac{\pi}{4}}^{x=\pi} \left(\frac{1}{2} - y\right) \mathrm{d}x$$

$$= \frac{1}{2}\left(\pi - \left(\frac{1}{\sqrt{2}} + \frac{\pi}{4}\right)\right) - \int_{x=\frac{1}{\sqrt{2}}+\frac{\pi}{4}}^{x=\pi} y \, \mathrm{d}x$$

$$= \left(\frac{3\pi}{8} - \frac{1}{2\sqrt{2}}\right) - \int_{\theta=\frac{\pi}{4}}^{\theta=\pi} \sin\theta\cos\theta(\cos\theta+1) \, \mathrm{d}\theta$$

$$= \frac{3\pi}{8} - \frac{1}{2\sqrt{2}} - \int_{\theta=\frac{\pi}{4}}^{\theta=\pi} \sin\theta\cos^2\theta + \sin\theta\cos\theta \, \mathrm{d}\theta$$

$$= \frac{3\pi}{8} - \frac{1}{2\sqrt{2}} - \left[-\frac{(\cos\theta)^3}{3}\right]_{\theta=\frac{\pi}{4}}^{\theta=\pi} - \left[-\frac{(\cos\theta)^2}{2}\right]_{\theta=\frac{\pi}{4}}^{\theta=\pi}$$

$$= \frac{3\pi}{8} - \frac{1}{2\sqrt{2}} + \frac{1}{3}\left[-1 - \frac{1}{2\sqrt{2}}\right] + \frac{1}{2}\left[1 - \frac{1}{2}\right]$$

$$= \frac{3\pi}{8} - \frac{1}{12} - \frac{\sqrt{2}}{3}$$

9(a) (i)

$$x = (ax+b)(x+1) + c\left(x^2+2x+2\right)$$

$$= (a+c)x^2 + (a+b+2c)x + (b+2c)$$

Comparing coefficients,

$$\begin{array}{ccccccccc} a & & & + & c & = & 0 \\ a & + & b & + & 2c & = & 1 \\ & & b & + & 2c & = & 0 \end{array}$$

Solving, we have $a=1, b=2, c=-1$.

<u>Alternative</u>

Sub $x=-1, -1 = 0 + c(1-2+2) \Rightarrow c = -1$

Compare coeff x^2 : $a+c=0 \Rightarrow a-1=0 \Rightarrow a=1$

Compare coeff x : $a+b+2c=1 \Rightarrow 1+b-2=1 \Rightarrow b=2$

$\therefore a=1, b=2, c=-1$

(ii)

$$\int \frac{x}{(x+1)\left(x^2+2x+2\right)}\,\mathrm{d}x$$

$$= \int \frac{(x+2)(x+1)-\left(x^2+2x+2\right)}{(x+1)\left(x^2+2x+2\right)}\,\mathrm{d}x$$

$$= \int \frac{x+2}{x^2+2x+2} - \frac{1}{x+1}\,\mathrm{d}x$$

$$= \int \frac{x+1}{x^2+2x+2} + \frac{1}{(x+1)^2+1} - \frac{1}{x+1}\,\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{2(x+1)}{x^2+2x+2}\,\mathrm{d}x + \int \frac{1}{(x+1)^2+1} - \frac{1}{x+1}\,\mathrm{d}x$$

$$= \frac{1}{2}\ln\left((x+1)^2+1\right) + \tan^{-1}(x+1) - \ln|x+1| + c$$

2. (a)

$$\int \sin 2x \sin 3x\ \mathrm{d}x = -\frac{1}{2}\int -2\sin 2x \sin 3x\ \mathrm{d}x$$

$$= -\frac{1}{2}\int \cos 5x - \cos x\ \mathrm{d}x$$

$$= -\frac{1}{2}\left(\frac{1}{5}\sin 5x - \sin x\right) + C$$

$$= -\frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$$

2. (b)(i)

$$\int (\ln x)^2 \, dx$$

$$= x(\ln x)^2 - \int \frac{2\ln x}{x} \cdot x \, dx$$

$$= x(\ln x)^2 - 2\left[\int \ln x \, dx\right]$$

$$= x(\ln x)^2 - 2\left[x\ln x - \int 1 \, dx\right]$$

$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

$$\int_1^e (\ln x)^2 dx = \left[x(\ln x)^2 - 2x\ln x + 2x\right]_1^e$$

$$= (e - 2e + 2e) - (2)$$

$$= e - 2$$

2. (b) (ii)

Volume of revolution

$$= \pi\int_0^{\mathrm{e}}\left(\frac{y}{\mathrm{e}}\right)^2 \mathrm{d}y - \pi\int_1^{\mathrm{e}}(\ln y)^2 \mathrm{d}y$$

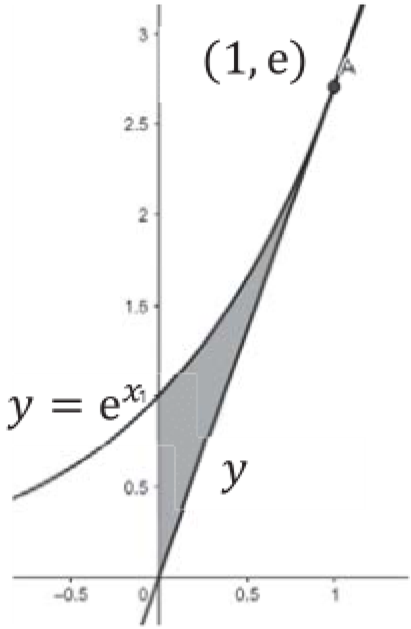
$$= \frac{\pi}{\mathrm{e}^2}\left[\frac{y^3}{3}\right]_0^{\mathrm{e}} - \pi(\mathrm{e}-2)$$

$$= \frac{1}{3}\pi\mathrm{e} - \pi(\mathrm{e}-2)$$

$$= \frac{1}{3}\pi\mathrm{e} - \pi\mathrm{e} + 2\pi$$

$$= 2\pi - \frac{2}{3}\pi\mathrm{e}$$

$$=2\pi\left(1-\frac{1}{3}\mathrm{e}\right)$$



3.

(a)(i)	$\tan(\theta+\theta)=\dfrac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}=\dfrac{2\tan\theta}{1-\tan^2\theta}$
(ii)	$x=\tan\theta \Rightarrow \dfrac{dx}{d\theta}=\sec^2\theta$ $x=0,\ \theta=0$ and $x=1,\ \theta=\dfrac{\pi}{4}$ $\displaystyle\int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right)\,dx=\int_0^{\frac{\pi}{4}}\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\left(\sec^2\theta\right)d\theta$ $\displaystyle=\int_0^{\frac{\pi}{4}}\tan^{-1}(\tan 2\theta)\left(\sec^2\theta\right)d\theta$ $\displaystyle=\int_0^{\frac{\pi}{4}}2\theta\sec^2\theta\ d\theta$ $\displaystyle=\left[2\theta\tan\theta\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}2\tan\theta\ d\theta$ $\displaystyle=2\left(\frac{\pi}{4}\right)-2\left[\ln\sec\theta\right]_0^{\frac{\pi}{4}}$ $=\dfrac{\pi}{2}-2\ln\sqrt{2}$ $=\dfrac{\pi}{2}-\ln 2$
(b)	$\displaystyle\int_{-1}^{1}\frac{2+\lvert x\rvert}{2+x}\,dx=\int_{-1}^{0}\frac{2-x}{2+x}\,dx+\int_0^1\frac{2+x}{2+x}\,dx$ $\displaystyle=\int_{-1}^{0}-1+\frac{4}{2+x}\,dx+\int_0^1 1\,dx$ $=\left[-x+4\ln\lvert 2+x\rvert\right]_{-1}^{0}+\left[x\right]_0^1$ $=4\ln 2$

Integration, volume, DE selected questions

4.

$$\frac{dx}{dt} = kx(5750 - x), \text{ where } k \text{ is a positive constant}$$

When $x = 750$, $\dfrac{dx}{dt} = 500$,

$$500 = k(750)(5750 - 750)$$

$$k = \frac{1}{7500}$$

$$\therefore \frac{dx}{dt} = \frac{1}{7500}x(5750 - x)$$

$$\frac{1}{x(5750 - x)}\frac{dx}{dt} = \frac{1}{7500}$$

$$\int \frac{1}{x(5750 - x)}\,dx = \int \frac{1}{7500}\,dt$$

$$\frac{1}{5750}\int \frac{1}{x} + \frac{1}{5750 - x}\,dx = \int \frac{1}{7500}\,dt$$

$$\ln|x| - \ln|5750 - x| = \frac{5750}{7500}t + c$$

$$\left|\frac{x}{5750 - x}\right| = e^{\frac{23}{30}t + c}$$

$$\frac{x}{5750 - x} = \pm e^{c} e^{\frac{23}{30}t}$$

$$= Ae^{\frac{23}{30}t}$$

When $t = 0$, $x = 750$, $\dfrac{750}{5000} = A \quad \Rightarrow \quad A = \dfrac{3}{20}$

$$x = \frac{17250}{3 + 20e^{-\frac{23}{30}t}}$$

5(b)(i)	$\dfrac{dn}{dt} = 50 - kn$ Since $\dfrac{dn}{dt} = 0$ when $n = 5000$, $0 = 50 - 5000k$ $k = 0.01$ $\dfrac{dn}{dt} = 50 - 0.01n$
5(b)(ii)	$\dfrac{dt}{dn} = \dfrac{1}{50 - 0.01n}$ $t = \displaystyle\int \frac{1}{50 - 0.01n}\,dn$ $= \displaystyle\int \frac{100}{5000 - n}\,dn$ $t = -100\ln\lvert 5000 - n\rvert + C'$ $\ln\lvert 5000 - n\rvert = -0.01t - C''$ $5000 - n = Ae^{-0.01t}$ where $A = \pm e^{-C''}$ $n = 5000 - Ae^{-0.01t}$ When $t = 0$, $n = 2000$: $A = 3000$

6.

(a)(i)	$y = 2te^n$ $\dfrac{dy}{dt} = 2e^n + 2te^n \dfrac{dn}{dt}$ Hence $e^n + te^n \dfrac{dn}{dt} = 4t$ $2e^n + 2te^n \dfrac{dn}{dt} = 8t$ $\dfrac{dy}{dt} = 8t$, where $k = 8$.
(a)(ii)	$\dfrac{dy}{dt} = 8t$ $y = 4t^2 + C$ Since $y = 2te^n$, $2te^n = 4t^2 + C$. Since $n = 0$ when $t = 1$, we have $2 = 4 + C \Rightarrow C = -2$. Hence

$$e^n = 2t - \frac{1}{t}$$

$$n = \ln\left(2t - \frac{1}{t}\right)$$

$$\therefore\ n = \ln\left(2t - \frac{1}{t}\right)$$