MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE
General Certificate of Education Advanced Level
Higher 2

MATHEMATICS

9740/01

Paper 1

October/November 2016

3 hours

Additional Materials: Answer Paper
Graph paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 Express $\dfrac{4x^2 + 4x - 14}{x - 4} - (x + 3)$ as a single simplified fraction. [2]

Hence, without using a calculator, solve the inequality

$$\frac{4x^2 + 4x - 14}{x - 4} < x + 3.$$ [3]

2 (i) Use your calculator to find the gradient of the curve $y = 2^{\cos x}$ at the points where $x = 0$ and $x = \frac{1}{2}\pi$. [2]

(ii) Find the equations of the tangents to this curve at the points where $x = 0$ and $x = \frac{1}{2}\pi$ and find the coordinates of the point where these tangents meet. [3]

3 The curve $y = x^4$ is transformed onto the curve with equation $y = \mathrm{f}(x)$. The turning point on $y = x^4$ corresponds to the point with coordinates (a, b) on $y = \mathrm{f}(x)$. The curve $y = \mathrm{f}(x)$ also passes through the point with coordinates $(0, c)$. Given that $\mathrm{f}(x)$ has the form $k(x - l)^4 + m$ and that a, b and c are positive constants with $c > b$, express k, l and m in terms of a, b and c. [2]

By sketching the curve $y = \mathrm{f}(x)$, or otherwise, sketch the curve $y = \dfrac{1}{\mathrm{f}(x)}$. State, in terms of a, b and c, the coordinates of any points where $y = \dfrac{1}{\mathrm{f}(x)}$ crosses the axes and of any turning points. [4]

4 An arithmetic series has first term a and common difference d, where a and d are non-zero. A geometric series has first term b and common ratio r, where b and r are non-zero. It is given that the 4th, 9th and 12th terms of the arithmetic series are equal to the 5th, 8th and 15th terms of the geometric series respectively.

(i) Show that r satisfies the equation $5r^{10} - 8r^3 + 3 = 0$. Given that $|r| < 1$, solve this equation, giving your answer correct to 2 decimal places. [4]

(ii) Using this value of r, find, in terms of b and n, the sum of the terms of the geometric series after, but not including, the nth term, simplifying your answer. [3]

5 The vectors $\mathbf{u}$ and $\mathbf{v}$ are given by $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = a\mathbf{i} + b\mathbf{k}$, where a and b are constants.

(i) Find $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$ in terms of a and b. [2]

(ii) Given that the $\mathbf{i}$- and $\mathbf{k}$-components of the answer to part **(i)** are equal, express $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$ in terms of a only. Hence find, in an exact form, the possible values of a for which $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$ is a unit vector. [4]

(iii) Given instead that $(\mathbf{u} + \mathbf{v}).(\mathbf{u} - \mathbf{v}) = 0$, find the numerical value of $|\mathbf{v}|$. [2]

6 ⑤ **(i)** Prove by the method of mathematical induction that

$$\sum_{r=1}^{n} r(r^2+1) = \tfrac{1}{4}n(n+1)(n^2+n+2). \qquad [5]$$

(ii) A sequence $u_0, u_1, u_2, \dots$ is given by

$$u_0 = 2 \quad \text{and} \quad u_n = u_{n-1} + n^3 + n \text{ for } n \geqslant 1.$$

Find u_1, u_2 and u_3. [2]

(iii) By considering $\sum_{r=1}^{n}(u_r - u_{r-1})$, find a formula for u_n in terms of n. [3]

7 **Do not use a calculator in answering this question.**

(a) Verify that $-1+5\mathrm{i}$ is a root of the equation $w^2 + (-1-8\mathrm{i})w + (-17+7\mathrm{i}) = 0$. Hence, or otherwise, find the second root of the equation in cartesian form, $p + \mathrm{i}q$, showing your working. [5]

(b) The equation $z^3 - 5z^2 + 16z + k = 0$, where k is a real constant, has a root $z = 1 + a\mathrm{i}$, where a is a positive real constant. Find the values of a and k, showing your working. [5]

8 It is given that $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \tan(ax+b)$ for constants a and b.

(i) Show that $\mathrm{f}'(x) = a + ay^2$. Use this result to find $\mathrm{f}''(x)$ and $\mathrm{f}'''(x)$ in terms of a and y. [5]

(ii) In the case where $b = \frac{1}{4}\pi$, use your results from part **(i)** to find the Maclaurin series for $\mathrm{f}(x)$ in terms of a, up to and including the term in x^3. [3]

(iii) Find the first two non-zero terms in the Maclaurin series for $\tan 2x$. [3]

[Questions 9, 10 & 11 are printed on the next page.]

9 A stone is held on the surface of a pond and released. The stone falls vertically through the water and the distance, x metres, that the stone has fallen in time t seconds is measured. It is given that $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

(i) The motion of the stone is modelled by the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} = 10.$$

(a) By substituting $y = \frac{dx}{dt}$, show that the differential equation can be written as $\frac{dy}{dt} = 10 - 2y$. [1]

(b) Find y in terms of t and hence find x in terms of t. [6]

(ii) A second model for the motion of the stone is suggested, given by the differential equation

$$\frac{d^2x}{dt^2} = 10 - 5\sin\tfrac{1}{2}t.$$

Find x in terms of t for this model. [3]

(iii) The pond is 5 metres deep. For each of these models, find the time the stone takes to reach the bottom of the pond, giving your answers correct to 2 decimal places. [2]

10 **(a)** The function f is given by $f : x \mapsto 1 + \sqrt{x}$, for $x \in \mathbb{R}, x \geqslant 0$.

(i) Find $f^{-1}(x)$ and state the domain of f^{-1}. [3]

(ii) Show that if $ff(x) = x$ then $x^3 - 4x^2 + 4x - 1 = 0$. Hence find the value of x for which $ff(x) = x$. Explain why this value of x satisfies the equation $f(x) = f^{-1}(x)$. [5]

(b) The function g, with domain the set of non-negative integers, is given by

$$g(n) = \begin{cases} 1 & \text{for } n = 0, \\ 2 + g(\tfrac{1}{2}n) & \text{for } n \text{ even}, \\ 1 + g(n-1) & \text{for } n \text{ odd}. \end{cases}$$

(i) Find g(4), g(7) and g(12). [3]

(ii) Does g have an inverse? Justify your answer. [2]

11 The plane p has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} a \\ 4 \\ -2 \end{pmatrix}$, and the line l has equation $\mathbf{r} = \begin{pmatrix} a-1 \\ a \\ a+1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$, where a is a constant and λ, μ and t are parameters.

(i) In the case where $a = 0$,

(a) show that l is perpendicular to p and find the values of λ, μ and t which give the coordinates of the point at which l and p intersect, [5]

(b) find the cartesian equations of the planes such that the perpendicular distance from each plane to p is 12. [5]

(ii) Find the value of a such that l and p do not meet in a unique point. [3]