MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE
General Certificate of Education Advanced Level
Higher 2

MATHEMATICS

9758/01

Paper 1

October/November 2017

3 hours

Additional Materials: Answer Paper
Graph paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number and name on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
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At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 Using standard series from the List of Formulae (MF26), expand $e^{2x}\ln(1+ax)$ as far as the term in x^3, where a is a non-zero constant. Hence find the value of a for which there is no term in x^2. [4]

2 (i) On the same axes, sketch the graphs of $y = \dfrac{1}{x-a}$ and $y = b|x-a|$, where a and b are positive constants. [2]

(ii) Hence, or otherwise, solve the inequality $\dfrac{1}{x-a} < b|x-a|$. [4]

3 **Do not use a calculator in answering this question.**

A curve C has equation $y^2 - 2xy + 5x^2 - 10 = 0$.

(i) Find the exact x-coordinates of the stationary points of C. [4]

(ii) For the stationary point with $x > 0$, determine whether it is a maximum or minimum. [3]

4 A curve C has equation $y = \dfrac{4x+9}{x+2}$.

(i) Show that the gradient of C is negative for all points on C. [3]

(ii) By expressing the equation of C in the form $y = a + \dfrac{b}{x+2}$, where a and b are constants, write down the equations of the asymptotes of C. [3]

(iii) Describe a pair of transformations which transforms the graph of C on to the graph of $y = \dfrac{1}{x}$. [2]

5 When the polynomial $x^3 + ax^2 + bx + c$ is divided by $(x-1)$, $(x-2)$ and $(x-3)$, the remainders are 8, 12 and 25 respectively.

(i) Find the values of a, b and c. [4]

A curve has equation $y = f(x)$, where $f(x) = x^3 + ax^2 + bx + c$, with the values of a, b and c found in part **(i)**.

(ii) Show that the gradient of the curve is always positive. Hence explain why the equation $f(x) = 0$ has only one real root and find this root. [3]

(iii) Find the x-coordinates of the points where the tangent to the curve is parallel to the line $y = 2x - 3$. [3]

6 (i) Interpret geometrically the vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and t is a parameter. [2]

(ii) Interpret geometrically the vector equation $\mathbf{r}.\mathbf{n} = d$, where $\mathbf{n}$ is a constant unit vector and d is a constant scalar, stating what d represents. [3]

(iii) Given that $\mathbf{b}.\mathbf{n} \neq 0$, solve the equations $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r}.\mathbf{n} = d$ to find $\mathbf{r}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ and d. Interpret the solution geometrically. [3]

7 It is given that $f(x) = \sin 2mx + \sin 2nx$, where m and n are positive integers and $m \neq n$.

(i) Find $\int \sin 2mx \sin 2nx \, dx$. [3]

(ii) Find $\int_0^{\pi} (f(x))^2 \, dx$. [5]

8 **Do not use a calculator in answering this question.**

(a) Find the roots of the equation $z^2(1 - i) - 2z + (5 + 5i) = 0$, giving your answers in cartesian form $a + ib$. [3]

(b) **(i)** Given that $\omega = 1 - i$, find ω^2, ω^3 and ω^4 in cartesian form. Given also that

$$\omega^4 + p\omega^3 + 39\omega^2 + q\omega + 58 = 0,$$

where p and q are real, find p and q. [4]

(ii) Using the values of p and q in part **(b)(i)**, express $\omega^4 + p\omega^3 + 39\omega^2 + q\omega + 58$ as the product of two quadratic factors. [3]

9 **(a)** A sequence of numbers $u_1, u_2, u_3, \ldots$ has a sum S_n where $S_n = \sum_{r=1}^{n} u_r$. It is given that $S_n = An^2 + Bn$, where A and B are non-zero constants.

(i) Find an expression for u_n in terms of A, B and n. Simplify your answer. [3]

(ii) It is also given that the tenth term is 48 and the seventeenth term is 90. Find A and B. [2]

(b) Show that $r^2(r + 1)^2 - (r - 1)^2 r^2 = kr^3$, where k is a constant to be determined. Use this result to find a simplified expression for $\sum_{r=1}^{n} r^3$. [4]

(c) D'Alembert's ratio test states that a series of the form $\sum_{r=0}^{\infty} a_r$ converges when $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, and diverges when $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$. When $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the test is inconclusive. Using the test, explain why the series $\sum_{r=0}^{\infty} \frac{x^r}{r!}$ converges for all real values of x and state the sum to infinity of this series, in terms of x. [4]

10 Electrical engineers are installing electricity cables on a building site. Points (x, y, z) are defined relative to a main switching site at $(0, 0, 0)$, where units are metres. Cables are laid in straight lines and the widths of cables can be neglected.

An existing cable C starts at the main switching site and goes in the direction $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$. A new cable is installed which passes through points $P(1, 2, -1)$ and $Q(5, 7, a)$.

(i) Find the value of a for which C and the new cable will meet. [4]

To ensure that the cables do not meet, the engineers use $a = -3$. The engineers wish to connect each of the points P and Q to a point R on C.

(ii) The engineers wish to reduce the length of cable required and believe in order to do this that angle PRQ should be $90°$. Show that this is not possible. [4]

(iii) The engineers discover that the ground between P and R is difficult to drill through and now decide to make the length of PR as small as possible. Find the coordinates of R in this case and the exact minimum length. [5]

11 Sir Isaac Newton was a famous scientist renowned for his work on the laws of motion. One law states that, for an object falling vertically in a vacuum, the rate of change of velocity, $v\,\mathrm{m\,s^{-1}}$, with respect to time, t seconds, is a constant, c.

(i) (a) Write down a differential equation relating v, t and c. [1]

(b) Initially the velocity of the object is $4\,\mathrm{m\,s^{-1}}$ and, after a further 2.5 s, the velocity of the object is $29\,\mathrm{m\,s^{-1}}$. Find v in terms of t and state the value of c. [3]

For an object falling vertically through the atmosphere, the rate of change of velocity is less than that for an object falling in a vacuum. The new rate of change of v is modelled as the difference between the value of c found in part **(i)(b)** and an amount proportional to the velocity v, with a constant of proportionality k.

(ii) Given that in this case the initial velocity is zero, find v in terms of t and k. [5]

For an object falling through the atmosphere, the 'terminal velocity' is the value approached by the velocity after a long time.

(iii) A falling object has initial velocity zero and terminal velocity $40\,\mathrm{m\,s^{-1}}$. Find how long it takes the object to reach 90% of its terminal velocity. [4]

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Section A: Pure Mathematics [40 marks]

1 A curve C has parametric equations

$$x = \frac{3}{t}, \qquad y = 2t.$$

(i) The line $y = 2x$ cuts C at the points A and B. Find the exact length of AB. [3]

(ii) The tangent at the point $P\left(\frac{3}{p}, 2p\right)$ on C meets the x-axis at D and the y-axis at E. The point F is the midpoint of DE. Find a cartesian equation of the curve traced by F as p varies. [5]

2 An arithmetic progression has first term 3. The sum of the first 13 terms of the progression is 156.

(i) Find the common difference. [2]

A geometric progression has first term 3 and common ratio r. The sum of the first 13 terms of the progression is 156.

(ii) Show that $r^{13} - 52r + 51 = 0$. Show that the common ratio cannot be 1 even though $r = 1$ is a root of this equation. Find the possible values of the common ratio. [4]

(iii) It is given that the common ratio of the geometric progression is positive, and that the nth term of this geometric progression is more than 100 times the nth term of the arithmetic progression. Write down an inequality, and hence find the smallest possible value of n. [3]

3 (a) The curve $y = \mathrm{f}(x)$ cuts the axes at $(a, 0)$ and $(0, b)$. It is given that $\mathrm{f}^{-1}(x)$ exists. State, if it is possible to do so, the coordinates of the points where the following curves cut the axes.

(i) $y = \mathrm{f}(2x)$

(ii) $y = \mathrm{f}(x - 1)$

(iii) $y = \mathrm{f}(2x - 1)$

(iv) $y = \mathrm{f}^{-1}(x)$

[4]

(b) The function g is defined by

$$\mathrm{g} : x \mapsto 1 - \frac{1}{1 - x}, \text{ where } x \in \mathbb{R},\ x \neq a.$$

(i) State the value of a and explain why this value has to be excluded from the domain of g. [2]

(ii) Find $\mathrm{g}^2(x)$ and $\mathrm{g}^{-1}(x)$, giving your answers in simplified form. [4]

(iii) Find the values of b such that $\mathrm{g}^2(b) = \mathrm{g}^{-1}(b)$. [2]

4 **(a)** A flat novelty plate for serving food on is made in the shape of the region enclosed by the curve $y = x^2 - 6x + 5$ and the line $2y = x - 1$. Find the area of the plate. [4]

(b) A curved container has a flat circular top. The shape of the container is formed by rotating the part of the curve $x = \dfrac{\sqrt{y}}{a - y^2}$, where a is a constant greater than 1, between the points $(0, 0)$ and $\left(\dfrac{1}{a-1}, 1\right)$ through 2π radians about the y-axis.

(i) Find the volume of the container, giving your answer as a single fraction in terms of a and π. [4]

(ii) Another curved container with a flat circular top is formed in the same way from the curve $x = \dfrac{\sqrt{y}}{b - y^2}$ and the points $(0, 0)$ and $\left(\dfrac{1}{b-1}, 1\right)$. It has a volume that is four times as great as the container in part **(i)**. Find an expression for b in terms of a. [3]

Section B: Probability and Statistics [60 marks]

5 A bag contains 6 red counters and 3 yellow counters. In a game, Lee removes counters at random from the bag, one at a time, until he has taken out 2 red counters. The total number of counters Lee removes from the bag is denoted by T.

(i) Find $P(T = t)$ for all possible values of t. [3]

(ii) Find $E(T)$ and $Var(T)$. [2]

Lee plays this game 15 times.

(iii) Find the probability that Lee has to take at least 4 counters out of the bag in at least 5 of his 15 games. [2]

6 A children's game is played with 20 cards, consisting of 5 sets of 4 cards. Each set consists of a father, mother, daughter and son from the same family. The family names are Red, Blue, Green, Yellow and Orange. So, for example, the Red family cards are father Red, mother Red, daughter Red and son Red.

The 20 cards are arranged in a row.

(i) In how many different ways can the 20 cards be arranged so that the 4 cards in each family set are next to each other? [2]

(ii) In how many different ways can the cards be arranged so that all five father cards are next to each other, all four Red family cards are next to each other and all four Blue family cards are next to each other? [3]

The cards are now arranged at random in a circle.

(iii) Find the probability that no two father cards are next to each other. [4]

7 The production manager of a food manufacturing company wishes to take a random sample of a certain type of biscuit bar from the thousands produced one day at his factory, for quality control purposes. He wishes to check that the mean mass of the bars is 32 grams, as stated on the packets.

(i) State what it means for a sample to be random in this context. [1]

The masses, x grams, of a random sample of 40 biscuit bars are summarised as follows.

$$n = 40 \qquad \Sigma(x-32) = -7.7 \qquad \Sigma(x-32)^2 = 11.05$$

(ii) Calculate unbiased estimates of the population mean and variance of the mass of biscuit bars. [2]

(iii) Test, at the 1% level of significance, the claim that the mean mass of biscuit bars is 32 grams. You should state your hypotheses and define any symbols you use. [5]

(iv) Explain why there is no need for the production manager to know anything about the population distribution of the masses of the biscuit bars. [2]

8 **(a)** Draw separate scatter diagrams, each with 8 points, all in the first quadrant, which represent the situation where the product moment correlation coefficient between variables x and y is

(i) -1,

(ii) 0,

(iii) between 0.5 and 0.9.

[3]

(b) An investigation into the effect of a fertiliser on yields of corn found that the amount of fertiliser applied, x, resulted in the average yields of corn, y, given below, where x and y are measured in suitable units.

x	0	40	80	120	160	200
y	70	104	118	119	126	129

(i) Draw a scatter diagram for these values. State which of the following equations, where a and b are positive constants, provides the most accurate model of the relationship between x and y.

(A) $y = ax^2 + b$ (B) $y = \dfrac{a}{x^2} + b$

(C) $y = a\ln 2x + b$ (D) $y = a\sqrt{x} + b$

[2]

(ii) Using the model you chose in part **(i)**, write down the equation for the relationship between x and y, giving the numerical values of the coefficients. State the product moment correlation coefficient for this model. [3]

(iii) Give two reasons why it would be reasonable to use your model to estimate the value of y when $x = 189$. [2]

9 On average 8% of a certain brand of kitchen lights are faulty. The lights are sold in boxes of 12.

(i) State, in context, two assumptions needed for the number of faulty lights in a box to be well modelled by a binomial distribution. [2]

Assume now that the number of faulty lights in a box has a binomial distribution.

(ii) Find the probability that a box of 12 of these kitchen lights contains at least 1 faulty light. [1]

The boxes are packed into cartons. Each carton contains 20 boxes.

(iii) Find the probability that each box in one randomly selected carton contains at least one faulty light. [1]

(iv) Find the probability that there are at least 20 faulty lights in a randomly selected carton. [2]

(v) Explain why the answer to part **(iv)** is greater than the answer to part **(iii)**. [1]

The manufacturer introduces a quick test to check if lights are faulty. Lights identified as faulty are discarded. If a light is faulty there is a 95% chance that the quick test will correctly identify the light as faulty. If the light is not faulty, there is a 6% chance that the quick test will incorrectly identify the light as faulty.

(vi) Find the probability that a light identified as faulty by the quick test is **not** faulty. [3]

(vii) Find the probability that the quick test correctly identifies lights as faulty or not faulty. [1]

(viii) Discuss briefly whether the quick test is worthwhile. [1]

10 A small component for a machine is made from two metal spheres joined by a short metal bar. The masses in grams of the spheres have the distribution $N(20, 0.5^2)$.

(i) Find the probability that the mass of a randomly selected sphere is more than 20.2 grams. [1]

In order to protect them from rusting, the spheres are given a coating which increases the mass of each sphere by 10%.

(ii) Find the probability that the mass of a coated sphere is between 21.5 and 22.45 grams. State the distribution you use and its parameters. [3]

(iii) The masses of the metal bars are normally distributed such that 60% of them have a mass greater than 12.2 grams and 25% of them have a mass less than 12 grams. Find the mean and standard deviation of the masses of metal bars. [4]

(iv) The probability that the total mass of a component, consisting of two randomly chosen coated spheres and one randomly chosen bar, is more than k grams is 0.75. Find k, stating the parameters of any distribution you use. [4]