1. **[IJC Prelims 17]**

The function f is given by $f : x \mapsto 3 + \dfrac{1}{x-2}$ for $x \in \mathbb{R}, x > 2$.

(a) Find $f^{-1}(x)$ and state the domain of f^{-1}. [3]

(b) Explain why the composite function f^2 exists. [1]

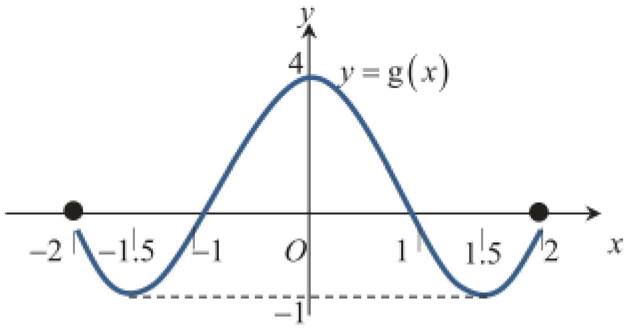
(c) Find the value of x for which $f^2(x) = x$. Explain why this value of x satisfies the equation $f(x) = f^{-1}(x)$. [3]

2. **[TPJC Prelims 17 (modified)]**

The function f is defined by

$$f : x \mapsto (x-k)^2, \quad x < k \text{ where } k > 5.$$

(a) Find $f^{-1}(x)$ and state the domain of f^{-1}. [3]



The diagram shows the curve with equation $y = g(x)$ with domain $D_g = [-2, 2]$. The curve crosses the x-axis at $x = -2, x = -1, x = 1$ and $x = 2$ and has turning points at $(-1.5, -1)$, $(0, 4)$ and $(1.5, -1)$.

(b) Explain why the composite function fg exists. [2]

(c) Find, in terms of k,

i. the value of $fg(-1)$, [1]

ii. the range of fg. [2]

3. **[TJC Prelims 18]**

The function f is defined by $f : x \mapsto (x^2 - 4)^2 - 24, x \in \mathbb{R}$.

(a) Sketch the graph of $y = f(x)$, indicating clearly all intercepts and stationary points. [2]

(b) Explain why f^{-1} does not exist. [1]

(c) The functions f_1 and f_2 are defined by

$$f_1 : x \mapsto (x^2 - 4)^2 - 24, x \in \mathbb{R}, x \leq k,$$
$$f_1 : x \mapsto (x^2 - 4)^2 - 24, x \in \mathbb{R}, x > k,$$

where k is a real number. State the range of values of k for which f_1^{-1} exists and f_2^{-1} does not exist. [1]

(d) Using the largest possible value of k found in (c), find f_1^{-1} in a similar form. [4]

4. The functions f and g are defined by

$$f : x \mapsto x^2 + 2x - 3 \qquad \text{for } x \in \mathbb{R}, x < b,$$
$$g : x \mapsto \frac{3x+2}{x-3}, \qquad \text{for } x \in \mathbb{R}, x \neq 3.$$

(a) Determine, with reason, whether f^{-1} exists when

i. $b = -2$, [1]

ii. $b = 2$. [1]

(b) For the value of b in (a) such that f^{-1} exists,

i. solve $f(x) = f^{-1}(x)$ **exactly**. [3]

ii. define f^{-1}, stating clearly its domain. [3]

(c) Determine, with reason, whether gf exists when $b = 0$. [1]

(d) Find an expression for $g^{-1}(x)$. [2]

Hence determine

i. $g^2(x)$. [1]

ii. $g^{2017}(8)$. [1]

5. **[CJC Prelims 18]**

The function f is defined by

$$f : x \mapsto x^2 + 4x - 5, \qquad \text{for } x \leq k, k \in \mathbb{R}.$$

(a) Find the largest exact value of k such that f^{-1} exists. For this value of k, define f^{-1} in a similar form. [4]

The function g is defined by

$$g : x \mapsto \begin{cases} 4 - x^2, & \text{for } 0 < x \leq 2 \\ 2x - 4, & \text{for } 2 < x \leq 4 \end{cases}$$

and that $g(x) = g(x+4)$ for all real values of x.

(b) Sketch the graph of $y = g(x)$ for $-1 < x \leq 7$. [3]

(c) Using the results in part (a) and (b), explain why the composite function $f^{-1}g$ exists and find the exact value of $f^{-1}g(6)$. [4]

6. **[SRJC Prelims 18]**

(a) The function f is defined by

$$f : x \mapsto x^2 - 2x - 8, \qquad x \in \mathbb{R}, x > k.$$

i. State the least value of k such that f^{-1} exists and find f^{-1} in a similar form. [3]

ii. Using the value of k found in (i), state the set of values of x such that $f^{-1}f(x) = ff^{-1}(x)$. [1]

(b) The functions g and h are defined by

$$g : x \mapsto \sqrt{x+41} + a, \qquad x \geq -41, a \in \mathbb{R},$$
$$h : x \mapsto x^2 + 10x - 16, \qquad x \in \mathbb{R}, x < -7.$$

i. Find the exact value of x for which $h^{-1}(x) = h(x)$. [3]

ii. Explain clearly why the composite function gh exists. [1]

iii. Find gh in the form $bx + c$, where B is a real constant and c is in terms of a. Explain your answers clearly. [2]

iv. State the exact range of gh in terms of a. [1]

7. **[ACJC Prelims 17 (modified)]**
The function f is defined by

$$f : x \mapsto \sin\left(x + \frac{1}{4}\right)\pi - \sin\left(x - \frac{3}{4}\right)\pi, \quad x \in \mathbb{R}, a \leq x \leq 1.$$

The function g is defined by

$$g : x \mapsto \frac{2x}{1-x}, \quad x \in \mathbb{R}, x \geq \frac{13}{5}.$$

(a) Express $f(x)$ as a single trigonometric function in the form $b\cos(x - c)\pi$. Hence state the range of f and sketch the curve when $a = -1$, labelling the exact coordinates of the points where the curve crosses the x- and y- axes. [4]

(b) State the least value of a such that f^{-1} exists, and define f^{-1} in similar form. [3]

(c) When $a = -\frac{13}{4}$, show that fg exists. Find the range of fg. [3]

8. **[AJC 17 Prelims]**

(a) The function f is defined by

$$f : x \mapsto \frac{e^x - 1}{e - 1} \quad \text{for } x \in \mathbb{R}.$$

Sketch the graph of $y = f(x)$ and state the range of f. [3]

(b) Another function h is defined by

$$h : x \mapsto \begin{cases} (x-1)^2 + 1 & \text{for } x \leq 1 \\ 1 - \frac{|1-x|}{2} & \text{for } 1 < x \leq 4 \end{cases}$$

Sketch the graph of $y = h(x)$ for $x \leq 4$ and explain why the composite function $f^{-1}h$ exists. Hence find the exact value of $(f^{-1}h)^{-1}(3)$. [7]

9. **[DHS Prelims 17 (modified)]**

(a) Express $\sin x + \sqrt{3}\cos x$ as $R\sin(x+\alpha)$, where $R > 0$ and α is an acute angle. [1]

The function f is defined by

$$f : x \mapsto \sin x + \sqrt{3}\cos x, \quad x \in \mathbb{R}, -\frac{\pi}{3} \le x \le \frac{\pi}{6}.$$

(b) Sketch the graph of $y = f(x)$. [2]

(c) Find $f^{-1}(x)$, stating the domain of f^{-1}. On the same diagram as in part (b), sketch the graph of $y = f^{-1}(x)$, indicating the equation of the line of symmetry. [4]

(d) ** Using integration, find the area of the region bounded by the graph of f^{-1} and the axes. [3]

The function g is defined by

$$g : x \mapsto |\ln(x+2)|, \quad \text{for } x \in \mathbb{R}, x > -2.$$

(e) Show that the composite function gf^{-1} exists, and find the range of gf^{-1}. [3]

10. **[NJC Prelims 13]**
The function f is defined by

$$f : x \mapsto \frac{cx - d}{dx - c}, \quad \text{for } x \in \mathbb{R}, x \neq \frac{c}{d},$$

where c and d are fixed constants and $c > d > 0$.

(a) Sketch the graph of $y = f(x)$, indicating the axial intercepts and the equations of any asymptotes. [3]

(b) Find $f^{-1}(x)$ and $f^2(x)$, stating clearly their domains. Hence, state the range of f^2. [5]

(c) On the same diagram in (a), sketch the graph of $y = f^2(x)$. Label your graph clearly. [1]

(d) State the value of $f^{2013}(1)$. [1]

Answers

1. (a) $f^{-1}(x) = 2 + \frac{1}{x-3}, x \in \mathbb{R}, x > 3$.
   (c) $x = 3.62$.

2. (a) $f^{-1}(x) = -\sqrt{x} + k$. $D_{f^{-1}} = (0, \infty)$.
   (b) $R_g = [-1, 4] \subseteq (-\infty, k) = D_f$ since $k > 5$. Hence fg exists.
   (c) i. k^2.
      ii. $[(4-k)^2, (1+k)^2]$.

3. (b) The line $y = 0$ cuts the graph of $y = f(x)$ more than once. Hence f is not one-one and f^{-1} does not exist.
   (c) $k \leq -2$.
   (d) $f_1^{-1} : x \mapsto -\sqrt{4 + \sqrt{x + 24}}, x \in \mathbb{R}, x \geq -24$.

4. (a) i. Yes. All horizontal lines $y = k, k \in \mathbb{R}$ cuts the curve $y = f(x)$ at most once. Hence f is a one-one function and f^{-1} exists.
      ii. No. The horizontal line $y = 0$ cuts the curve $y = f(x)$ more than once. Hence f is not a one-one function and f^{-1} does not exist.
   (b) i. $\frac{-1-\sqrt{13}}{2}$.
      ii. $f^{-1} : x \mapsto -1 - \sqrt{x + 4}, x \in \mathbb{R}, x > -3$.
   (c) $R_f = (-3, \infty) \nsubseteq D_g = (-\infty, 3) \cup (3, \infty)$.
   (d) $g^{-1}(x) = \frac{3x+2}{x-3}$.
      i. x.
      ii. $\frac{26}{5}$.

5. (a) $f^{-1} : x \mapsto -2 - \sqrt{x + 9}, \quad$ for $x \geq -9$.
   (c) $[0, 4] = R_g \subseteq D_{f^{-1}} = [-9, \infty)$.
   $f^{-1}g(6) = -5$.

6. (a) i. Least $k = 1$.
      $f^{-1} : x \mapsto 1 + \sqrt{x + 9}, x \in \mathbb{R}, x > -9$.
      ii. $(1, \infty)$.
   (b) i. $-\frac{9}{2} - \frac{\sqrt{145}}{2}$.
      ii. $(-37, \infty) = R_h \subseteq D_g = [-41, \infty)$.
      iii. $-x + a - 5$.
      iv. $(a + 2, \infty)$.

7. (a) $b = 2, c = \frac{1}{4}$.
   $R_f = [-2, 2]$, $(-\frac{1}{4}, 0), (\frac{3}{4}, 0), (0, \sqrt{2})$.
   (b) $a = \frac{1}{4}, f^{-1} : x \mapsto \frac{1}{\pi} \cos^{-1} \frac{x}{2} + \frac{1}{4}, x \in [-\sqrt{2}, 2]$.
   (c) $R_{fg} = [-2, \sqrt{2})$.

8. $1 - \sqrt{\mathrm{e}^2 + \mathrm{e}}$.

9. (a) $2\sin(x+\frac{\pi}{3})$.

   (c) $f^{-1}(x) = -\frac{\pi}{3} + \sin^{-1}(\frac{x}{2})$. $D_{f^{-1}} = R_f = [0, 2]$.

   (d) 1.

   (e) $R_{gf^{-1}} = [0, 0.926]$.

10. (b) $f^{-1}(x) = \frac{cx-d}{dx-c}$, $D_{f^{-1}} = (-\infty, \frac{c}{d}) \cup (\frac{c}{d}, \infty)$.
    $f^2(x) = x$, $D_{f^2} = (-\infty, \frac{c}{d}) \cup (\frac{c}{d}, \infty)$.

    (c) $f^{2013}(1) = -1$.