1 At the start of the year, Mr Toh invested in three types of savings bonds, namely "Ucare", "Ushare" and "Ugain". The amount invested in "Ugain" is equal to the sum of the amounts invested in the other 2 bonds. In addition, the sum of the amount invested in "Ushare" and twice the amount invested in "Ugain" is 8 times the amount invested in "Ucare".

At the end of the year, "Ucare", "Ushare" and "Ugain" paid out interest at a rate of 2.5%, 1.75% and 3% respectively. Mr Toh received a total of $657.60 in interest.

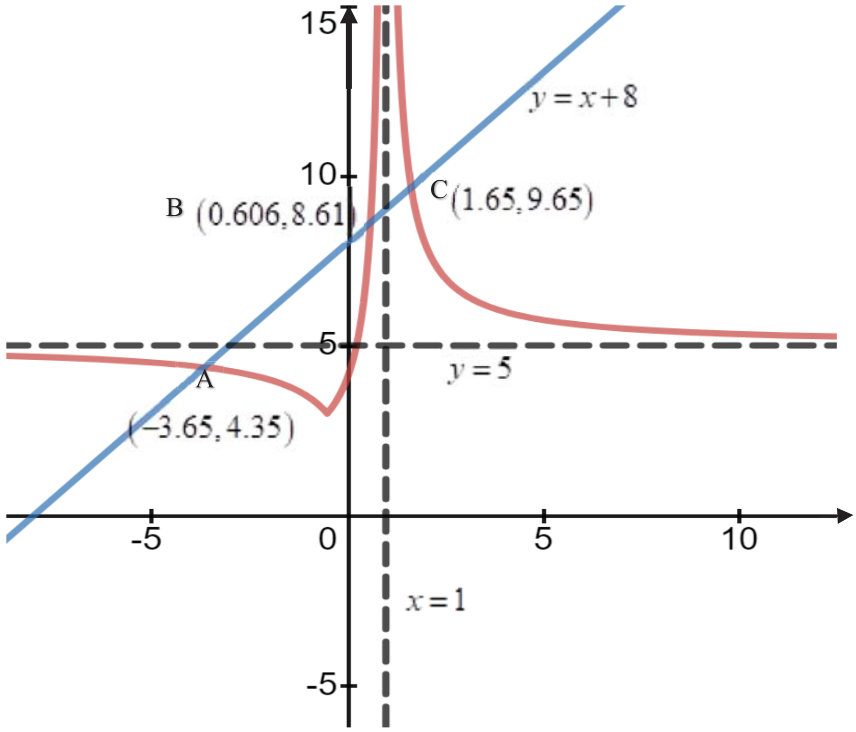
Express this information as 3 linear equations and hence find the amount invested in savings bond "Ushare". [4]

Suggested solution
Let *c*, *s* and *g* be the amount of investments in saving bonds 'Ucare', 'Ushare' and 'Ugain' respectively. **Amount invested in bond 'Ugain' equals the sum of the amounts invested in the other 2 bonds:** $c+s=g$ $c+s-g=0$ **----- (1)** **The sum of the amount invested in "Ushare" and twice the amount invested in "Ugain" is 8 times the amount invested in "Ucare".** $s+2g=8c$ $8c-s-2g=0$ **-----(2)** **Total interest of $657.60:** $0.025c+0.0175s+0.03g=657.60$ **----(3)** **Solving simultaneously,** $c=\$4384$, $s=\$8768$ and $g=\$13152$ **Total amount of investments for Ushare is $8768**

2 Sketch the curve $y = 3 + \left|\frac{2x+1}{x-1}\right|$, stating the equations of the asymptotes.

Hence, solve the inequality $\left|\frac{2x+1}{x-1}\right| > x+5$. [5]

Suggested solution



Note that since $y = \frac{2x+1}{x-1} = 2 + \frac{3}{x-1}$, the horizontal asymptote is given by $y = 5$.

$$\left|\frac{2x+1}{x-1}\right| > x+5 \Rightarrow 3 + \left|\frac{2x+1}{x-1}\right| > x+8$$

Sketch the graph of $y = x+8$ on the same axes.

By GC, the intersection points are given by:

$A\ (-3.65, 4.35)$

$B\ (0.606, 8.61)$

$C\ (1.65, 9.65)$

Therefore, the solution is:

$x < -3.65$ or $0.606 < x < 1$ or $1 < x < 1.65$

(Note that $x = 1$ is not part of the solution)

3 The curve C has equation $a^2x^2 - y^2 - 2(ax - 2y + 2) = 0$ where a is a constant such $a \in \mathbb{R} \setminus \{0\}$.

Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a(ax-1)}{y-2}$.

Hence, find the coordinates of the points on C at which the tangent is parallel to y-axis. [6]

Suggested solution

$$a^2x^2 - y^2 - 2(ax - 2y + 2) = 0$$

$$2a^2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2(a - 2\frac{\mathrm{d}y}{\mathrm{d}x}) = 0$$

$$2a^2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2a + 4\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(2 - y) = a - a^2x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a - a^2x}{2 - y} = \frac{a(ax-1)}{y-2}$$

The point on C at which the tangent is parallel to y-axis

$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = 0$$

$$\Rightarrow 2 - y = 0$$
$$\Rightarrow y = 2$$

$$a^2x^2 - y^2 - 2(ax - 2y + 2) = 0$$

$$a^2x^2 - 2^2 - 2(ax - 2(2) + 2) = 0$$
$$a^2x^2 - 2ax = 0$$
$$ax[ax - 2] = 0$$
$$x = \frac{2}{a} \quad \text{or} \quad x = 0$$

$\therefore (0, 2)$ and $\left(\frac{2}{a}, 2\right)$ are the two points which its tangent is parallel to to the y-axis.

4 Referred to the origin O, the points A, B, and C have position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ respectively.

(i) Given that non-zero numbers λ and μ are such that $\lambda\mathbf{a} + \mu\mathbf{b} + \mathbf{c} = \mathbf{0}$ and $\lambda + \mu + 1 = 0$ with $\mu > 0$. Show that A, B, and C are collinear and find the ratio $CA : CB$ in terms of μ. [4]

(ii) F is another point such that the line passing through A, B, and C does not contain it. Find $\dfrac{\left|\overrightarrow{BF} \times \overrightarrow{BC}\right|}{\left|\overrightarrow{AF} \times \overrightarrow{AC}\right|}$ in terms of μ. [2]

Suggested solution

(i)

<u>Method 1</u>

We first show A, B, and C are collinear by $\overrightarrow{CA} = k\,\overrightarrow{CB}$, where k is a constant to be found in terms of μ, and note that we can then also use the value of constant k to determine the ratio CA: CB later.

Using $\lambda\mathbf{a} + \mu\mathbf{b} + \mathbf{c} = \mathbf{0}$, **and** $\lambda + \mu + 1 = 0$

$$\begin{aligned}\overrightarrow{CA} &= \mathbf{a} - \mathbf{c} \\ &= \mathbf{a} + \lambda\mathbf{a} + \mu\mathbf{b} \\ &= \mathbf{a} + (-\mu - 1)\mathbf{a} + \mu\mathbf{b} \\ &= \mu(\mathbf{b} - \mathbf{a})\end{aligned}$$

$$\begin{aligned}\overrightarrow{CB} &= \mathbf{b} - \mathbf{c} \\ &= \mathbf{b} + \lambda\mathbf{a} + \mu\mathbf{b} \\ &= \mathbf{b} + (-\mu - 1)\mathbf{a} + \mu\mathbf{b} \\ &= (\mu + 1)(\mathbf{b} - \mathbf{a})\end{aligned}$$

Thus, $\overrightarrow{CA} = \dfrac{\mu}{\mu + 1}\overrightarrow{CB}$,

since $\overrightarrow{CA}$ is a scalar multiple of $\overrightarrow{CB}$, and C is a common point, so A, B, and C are collinear.

$$\left|\overrightarrow{CA}\right| = \frac{\mu}{\mu + 1}\left|\overrightarrow{CB}\right|$$

$$\frac{CA}{CB} = \frac{\mu}{\mu + 1}$$

Thus $CA : CB = \mu : 1 + \mu$

<u>Method 2</u>

$(-\mu - 1)\,\mathbf{a} + \mu\mathbf{b} + \mathbf{c} = 0 \quad \because \lambda = -\mu - 1$

$\mu\,(\mathbf{b} - \mathbf{a}) + (\mathbf{c} - \mathbf{a}) = 0$

$\mu\overrightarrow{AB} + \overrightarrow{AC} = 0$

$\overrightarrow{AB} = -\dfrac{1}{\mu}\overrightarrow{AC}$

$\Rightarrow$ Since $\overrightarrow{AB}$ is a scalar multiple of $\overrightarrow{AC}$, and A is a common point, A, B, and C are collinear.

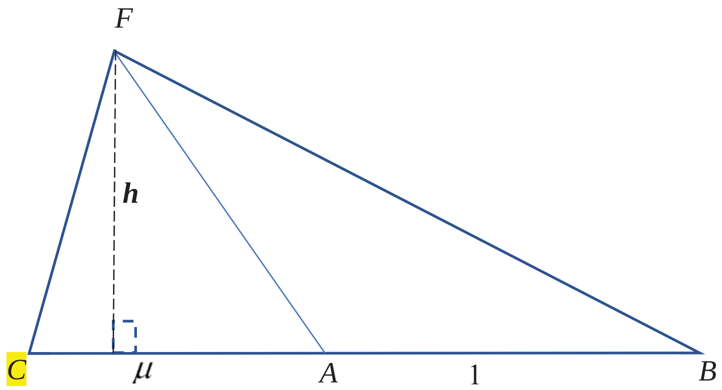
$$(-\mu-1)\mathbf{a}+\mu\mathbf{b}+\mathbf{c}=0$$

$$\mathbf{a}=\frac{-\mu\mathbf{b}-\mathbf{c}}{-\mu-1}$$

$$\mathbf{a}=\frac{\mu\mathbf{b}+\mathbf{c}}{\mu+1}$$

Thus, A divides BC in the ratio $1:\mu \Rightarrow CA:CB=\mu:1+\mu$

(ii)



$$\frac{\left|\overrightarrow{BF}\times\overrightarrow{BC}\right|}{\left|\overrightarrow{AF}\times\overrightarrow{AC}\right|}=\frac{\frac{1}{2}\times\left|\overrightarrow{BF}\times\overrightarrow{BC}\right|}{\frac{1}{2}\times\left|\overrightarrow{AF}\times\overrightarrow{AC}\right|}$$

$$=\frac{\text{area }\Delta BFC}{\text{area }\Delta AFC}$$

$$=\frac{1+\mu}{\mu}\quad(\because\text{ the two triangles have the same height.})$$

Method II:

Consider the fixed height $\boldsymbol{h}$:

$$\boldsymbol{h}=\frac{\left|\overrightarrow{BF}\times\overrightarrow{BC}\right|}{\left|\overrightarrow{BC}\right|}=\frac{\left|\overrightarrow{AF}\times\overrightarrow{AC}\right|}{\left|\overrightarrow{AC}\right|}$$

$$\Rightarrow\frac{\left|\overrightarrow{BF}\times\overrightarrow{BC}\right|}{\left|\overrightarrow{AF}\times\overrightarrow{AC}\right|}=\frac{\left|\overrightarrow{BC}\right|}{\left|\overrightarrow{AC}\right|}=\frac{1+\mu}{\mu}\text{ (from (i))}$$

5 **(a)** Find $\sum_{r=n}^{2n}\left(3^r+n\right)$. [3]

(b) Express $\ln\left(\frac{r^2}{r^2-1}\right)$ as $A\ln(r-1)+B\ln r+C\ln(r+1)$, where A, B and C are integers to be determined.

Hence, find $\sum_{r=2}^{n}\ln\left(\frac{r^2}{r^2-1}\right)$, leaving your answer as a single logarithmic function in terms of n. [4]

Suggested solution

$$\sum_{r=n}^{2n}\left(3^r+n\right)=\sum_{r=n}^{2n}3^r+\sum_{r=n}^{2n}n$$

$$=\frac{3^n\left(3^{n+1}-1\right)}{2}+n(n+1)$$

$$=\frac{3^n\left(3^{n+1}-1\right)+2n(n+1)}{2}$$

$$\ln\left(\frac{r^2}{r^2-1}\right)=\ln\left(\frac{r^2}{(r-1)(r+1)}\right)$$

$$=\ln r^2-\ln(r-1)-\ln(r+1)$$

$$=-\ln(r-1)+2\ln r-\ln(r+1)$$

$$\sum_{r=2}^{n}\ln\left(\frac{r^2}{r^2-1}\right)=\sum_{r=2}^{n}\left[-\ln(r-1)+2\ln r-\ln(r+1)\right]$$

$$\begin{aligned}
=[&-\ln 1+2\ln 2-\ln 3\\
&-\ln 2+2\ln 3-\ln 4\\
&-\ln 3+2\ln 4-\ln 5\\
&-\ln 4+2\ln 5-\ln 6\\
&+\ldots\\
&-\ln(n-3)+2\ln(n-2)-\ln(n-1)\\
&-\ln(n-2)+2\ln(n-1)-\ln(n)\\
&-\ln(n-1)+2\ln(n)-\ln(n+1)]
\end{aligned}$$

$$=-\ln 1+\ln 2+\ln(n)-\ln(n+1)$$

$$=\ln\left(\frac{2n}{n+1}\right)$$

6 The sum, S_n, of the first n terms of a sequence $\{u_n\}$ is given by $S_n = n^2 - (k-2)n$, where k is a non-zero real constant.

(i) Prove that the sequence $\{u_n\}$ is an arithmetic sequence. [3]

(ii) Given that u_8, u_4 and u_2 are the first 3 terms in a geometric sequence, find the value of k. [2]

(iii) Give a reason why the geometric series converges and find the value of the sum to infinity. [2]

	Suggested solution
(i)	$S_n = n^2 - (k-2)n$ $u_n = S_n - S_{n-1}$ $= n^2 - (k-2)n - \left[(n-1)^2 - (k-2)(n-1)\right]$ $= n^2 - kn + 2n - \left[n^2 - 2n + 1 - kn + 2n + k - 2\right]$ $= 2n + 1 - k$ $u_n - u_{n-1}$ $= 2n + 1 - k - \left[2(n-1) + 1 - k\right]$ $= 2$ is a constant independent of n $\therefore$ the sequence $\{u_n\}$ is an arithmetic sequence
(ii)	$u_8 = 2(8) + 1 - k = 17 - k$ $u_4 = 2(4) + 1 - k = 9 - k$ $u_2 = 2(2) + 1 - k = 5 - k$ $\dfrac{9-k}{17-k} = \dfrac{5-k}{9-k}$ $(9-k)^2 = (17-k)(5-k)$ $81 - 18k + k^2 = 85 - 22k + k^2$ $4k - 4 = 0$ $k = 1$ **Alternative Solution:** Given that u_n is a AP, so $u_n = a + 2(n-1)$, where a is the first term in the AP.

$u_8 = a + 2(7) = a + 14$
$u_4 = a + 2(3) = a + 6$
$u_2 = a + 2(1) = a + 2$

Since $u_8,\ u_4,\ u_2$ are first three terms in a GP,

$$\frac{a+6}{a+14} = \frac{a+2}{a+6}$$
$$(a+6)^2 = (a+2)(a+14)$$
$$a^2 + 12a + 36 = a^2 + 16a + 28$$
$$4a = 8$$
$$a = 2$$

$$S_1 = a$$
$$1^2 - (k-2)(1) = 2$$
$$k = 1$$

(iii) $r = \dfrac{9-k}{17-k} = \dfrac{8}{16} = \dfrac{1}{2}$

Since $|r| = \dfrac{1}{2} < 1$, the geometric series is convergent and

$$\therefore S_\infty = \frac{u_8}{1-r} = \frac{16}{1-\frac{1}{2}} = 32$$

7 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{bx}{ax-b}, \quad \text{for } x \in \mathbb{R}, x \neq \frac{b}{a},$$

where a and b are non-zero constants.

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(ii) Hence show that $\mathrm{f}^2(x) = x$, and write down $\mathrm{f}^n(x)$ where n is an odd number. [2]

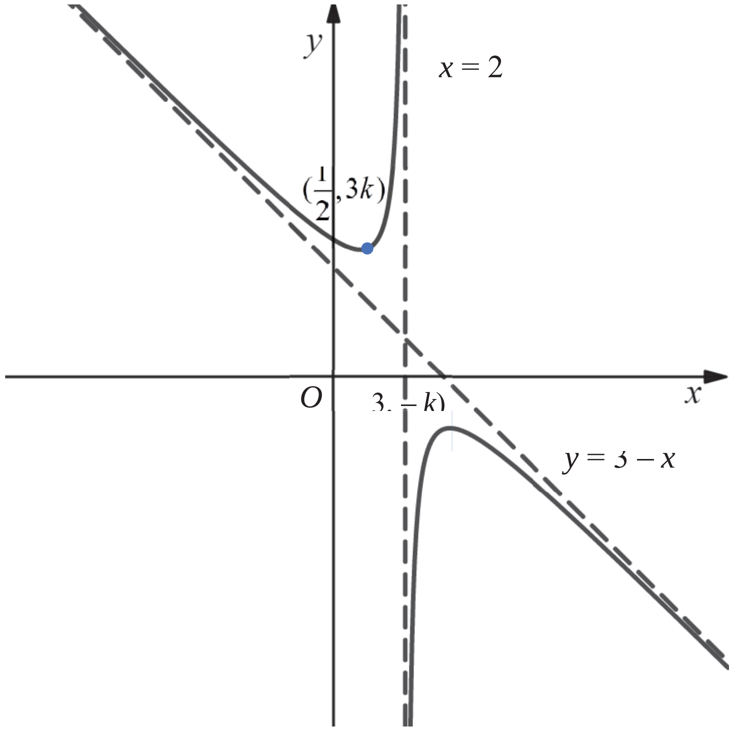
The function g is defined by $\mathrm{g}: x \mapsto 2+\mathrm{e}^{-x}$, for $x \in \mathbb{R}, x \geq 0$. If $a=2$ and $b=1$ for function f,

(iii) explain why the composite function gf does not exist. [1]

(iv) find an expression for $\mathrm{fg}(x)$ and state the domain and exact range of fg . [3]

Suggested solution	
(i)	$\mathrm{f}: x \mapsto \frac{bx}{ax-b}, \quad \text{for } x \in \mathbb{R}, x \neq \frac{b}{a},$ Let $y = \frac{bx}{ax-b} \Rightarrow y(ax-b) = bx$ $\Rightarrow axy - bx = by$ $\Rightarrow x = \frac{by}{ay-b}$ $\mathrm{f}^{-1}: x \mapsto \frac{bx}{ax-b}, \quad \text{for } x \in \mathbb{R}, x \neq \frac{b}{a},$
(ii)	Since $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, $\mathrm{f}^2(x) = \mathrm{f}\left[\mathrm{f}^{-1}(x)\right] = x$ $\mathrm{f}^3(x) = \mathrm{f}\left[\mathrm{f}^2(x)\right] = \mathrm{f}(x)$, therefore for $\mathrm{f}^n(x) = \mathrm{f}(x) = \frac{bx}{ax-b}$ for n odd,
(iii)	If $a=2$ and $b=1$ for function f, $\mathrm{f}: x \mapsto \frac{x}{2x-1}, \quad \text{for } x \in \mathbb{R}, x \neq \frac{1}{2},$ $\mathrm{R_f} = \mathbb{R} \setminus \left\{\frac{1}{2}\right\}$ $\mathrm{D}_g = [0, \infty)$ As $\mathrm{R_f} \not\subset \mathrm{D}_g$, gf does not exist.
(iv)	$\mathrm{fg}(x) = \mathrm{f}\left(2+\mathrm{e}^{-x}\right)$ $= \frac{2+\mathrm{e}^{-x}}{2\left(2+\mathrm{e}^{-x}\right)-1}$ $= \frac{2+\mathrm{e}^{-x}}{3+2\mathrm{e}^{-x}}, \quad x \geq 0$ $\mathrm{D_{fg}} = \mathrm{D_g} = [0, \infty)$ Range of $\mathrm{R_{fg}} = \left[\frac{3}{5}, \frac{2}{3}\right)$

8 The diagram shows the curve $y = \mathrm{f}(x)$. The curve has a minimum point $(\frac{1}{2}, 3k)$ and a maximum point $(3, -k)$ where $k > 0$. It has asymptotes $x = 2$ and $y = 3 - x$.



Sketch, on separate diagrams, the graph of

(i) $y = \mathrm{f}(x+2) + k$, [3]

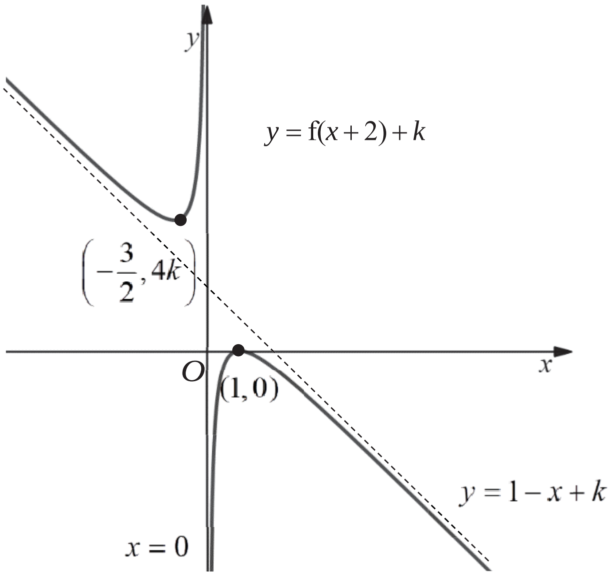
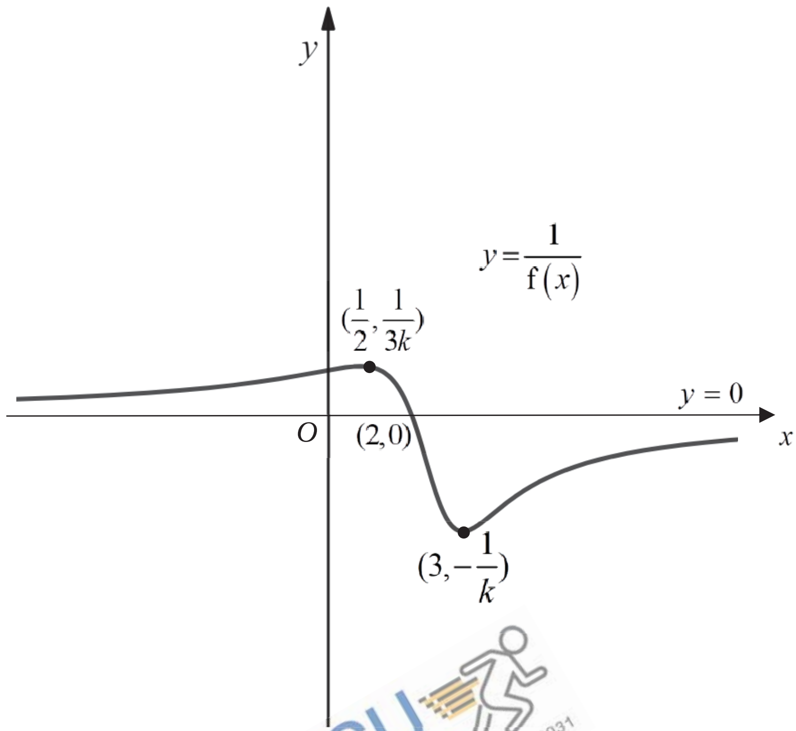
showing clearly the stationary points, axial intercepts and equations of all asymptotes.

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

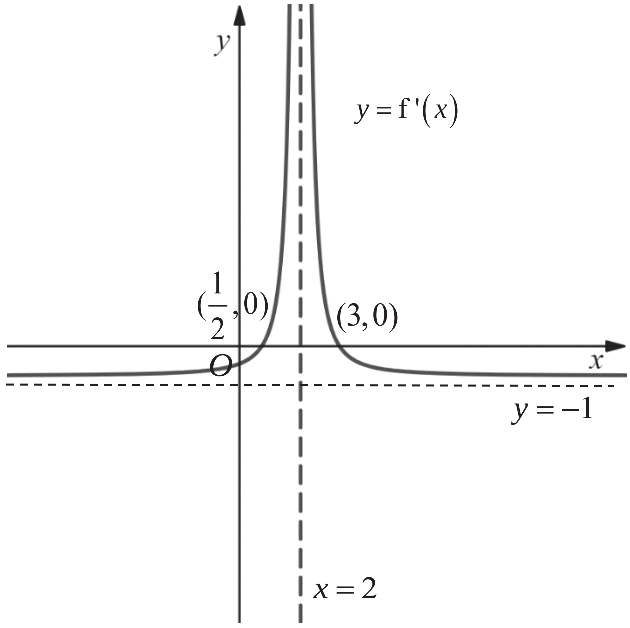
showing clearly the stationary points, axial intercepts and equations of all asymptotes.

(iii) $y = \mathrm{f}'(x)$, [3]

showing clearly the stationary points, axial intercepts and equations of all asymptotes.

Suggested solution
(i) 
(ii) 

(iii)



9 **(a)** Find $\displaystyle\int \frac{1-4x}{4x^2+1}\,\mathrm{d}x$. [4]

(b) Use the substitution $u=\sqrt{x}$ to show that $\displaystyle\int_0^{\frac{\pi^2}{4}} \sqrt{x}\sin\sqrt{x}\,\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} u^2 \sin u\,\mathrm{d}u$. [2]

Hence, evaluate the integral $\displaystyle\int_0^{\frac{\pi^2}{4}} \sqrt{x}\sin\sqrt{x}\,\mathrm{d}x$ exactly. [4]

Suggested solution

(a) $$\int \frac{1-4x}{4x^2+1}\,\mathrm{d}x = \int \left(\frac{1}{4x^2+1}-\frac{4x}{4x^2+1}\right)\mathrm{d}x$$

$$=\frac{1}{2}\int \frac{2}{(2x)^2+1}\,\mathrm{d}x-\frac{1}{2}\int \frac{8x}{4x^2+1}\,\mathrm{d}x$$

$$=\frac{1}{2}\tan^{-1}2x-\frac{1}{2}\ln\left(4x^2+1\right)+C,$$

where C is an arbitrary constant.

Alternative:

$$\int \frac{1}{4x^2+1}\,\mathrm{d}x = \frac{1}{4}\int \frac{1}{x^2+\left(\frac{1}{2}\right)^2}\,\mathrm{d}x = \frac{1}{4}\left(\frac{1}{\frac{1}{2}}\right)\tan^{-1}\frac{x}{\frac{1}{2}}$$

$$=\frac{1}{2}\tan^{-1}2x+C$$

(b) Given substitution: Let $u=\sqrt{x} \Rightarrow u^2=x$

When $x=0, u=0$ $\quad x=\dfrac{\pi^2}{4}, u=\dfrac{\pi}{2}$

Differentiating with respect to x, we have $2u\dfrac{\mathrm{d}u}{\mathrm{d}x}=1$

$$\therefore \frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2u} \quad \left(\text{or } \frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}\right)$$

Hence, $\displaystyle\int_0^{\frac{\pi^2}{4}} \sqrt{x}\sin\sqrt{x}\,\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} u^2\sin u\,\mathrm{d}u$

$$2\int_0^{\frac{\pi}{2}} u^2\sin u\,\mathrm{d}u = 2\left[u^2(-\cos u)\right]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} 2u(-\cos u)\,\mathrm{d}u \text{ (By parts)} =$$

$$2\left[-u^2\cos u\right]_0^{\frac{\pi}{2}} + 4\int_0^{\frac{\pi}{2}} u\cos u\,\mathrm{d}u \textbf{ (simplify)}$$

$$= 2\left[-u^2\cos u\right]_0^{\frac{\pi}{2}} + 4\left[\left[u\sin u\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\sin u\,\mathrm{d}u\right] =$$

$$2\left[-u^2\cos u+2u\sin u+2\cos u\right]_0^{\frac{\pi}{2}} =$$

$$2\left(\left(0+2\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2}+0\right)-(0+0+2)\right) = 2(\pi-2)$$

10 The curve C is defined parametrically by the equations $x = 2\operatorname{cosec}\theta$, $y = 2\cot\theta$, where $0 < \theta < \frac{\pi}{2}$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \sec\theta$. [2]

(ii) Find the equation of the tangent to the curve at the point $P(2\operatorname{cosec} p, 2\cot p)$ and show that the equation of the normal to the curve at P is $y = 4\cot p - x\cos p$. [3]

(iii) The tangent and normal at P cut the x-axis at T and N respectively.

Show that T has coordinates ($2\sin p$, 0).

By finding the x-coordinates of N in the simplest form, evaluate $|ON||OT|$. [4]

(iv) Find the rate of change of y at $(2\sqrt{2}, 2)$, given that x is increasing at a constant rate of 1 unit per second. [3]

Suggested solution	
(i) $x = 2\operatorname{cosec}\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -2\operatorname{cosec}\theta\cot\theta$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x}$ $= \frac{-2\operatorname{cosec}^2\theta}{-2\operatorname{cosec}\theta\cot\theta}$ $= \operatorname{cosec}\theta\tan\theta$ $= \frac{1}{\sin\theta}\frac{\sin\theta}{\cos\theta}$ $= \sec\theta$	$y = 2\cot\theta$ $\frac{\mathrm{d}y}{\mathrm{d}\theta} = -2\operatorname{cosec}^2\theta$
(ii)Equation of tangent at point P $(2\operatorname{cosec} p, 2\cot p)$ with $\theta = p$: $y - 2\cot p = \sec p(x - 2\operatorname{cosec} p)$ $y = 2(\cot p - \operatorname{cosec} p\sec p) + x\sec p$ Equation of normal at point P $(2\operatorname{cosec} p, 2\cot p)$ with $\theta = p$: $y - 2\cot p = -\frac{1}{\sec p}(x - 2\operatorname{cosec} p)$ $y = 2\cot p - (\cos p)\left(x - \frac{2}{\sin p}\right)$ $y = 4\cot p - x\cos p$	
(iii)At point T, $y = 0$ $0 = 2(\cot p - \operatorname{cosec} p\sec p) + x\sec p$	

$$x = \frac{2(\cot p - \operatorname{cosec} p \sec p)}{-\sec p}$$
$$= -2\ (\cos^2 p\ \operatorname{cosec} p - \operatorname{cosec} p)$$
$$= 2\operatorname{cosec} p(1 - \cos^2 p)$$
$$= 2\operatorname{cosec} p(\sin^2 p)$$
$$= 2\sin p$$

At point N, $y = 0$

$$0 = 4\cot p - x\cos p$$

$$x = \frac{4\cot p}{\cos p}$$
$$= 4\operatorname{cosec} p$$

$$|ON||OT| = |2\sin t||4\ \operatorname{cosec}\ t|$$
$$= 8$$

(iv) $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$

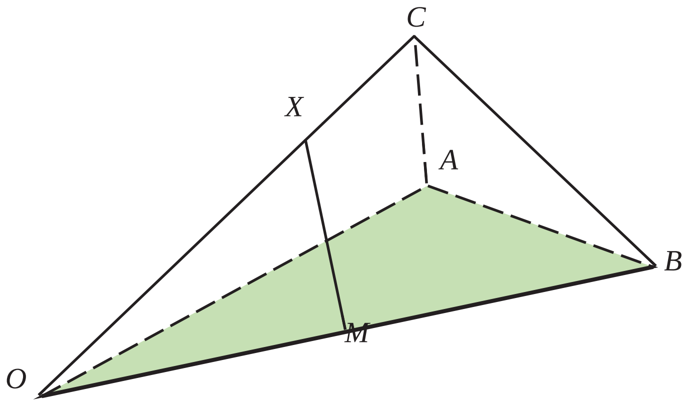
$$= (\sec\theta)(1)$$

At $(x, y) = (2\sqrt{2}, 2)$, $\theta = \frac{\pi}{4}$,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sec\frac{\pi}{4} = \sqrt{2}.$$

11 Engineers are building a greenhouse on an inclined grass slope. Points (x, y, z) are defined relative to a base point $O\,(0,0,0)$, where units are in metres.

The greenhouse is in the shape of a tetrahedron. The corners *O*, *A* and *B* lie on the grass slope, while the point *C* is the highest point of the tetrahedron (also called apex), as shown in the diagram below.



The coordinates of *A*, *B* and *C* are $(100,100,10)$, $(160,20,8)$ and $(80,59,20)$ respectively.

(i) Show that the Cartesian equation of the grass slope is $3x+4y-70z=0$. [2]

(ii) Find the angle of inclination of the grass slope. [2]

(iii) To secure the greenhouse, the architect plans to build a central pillar from the apex directly to the floor of the greenhouse such that the central pillar is perpendicular to the horizontal plane. Find the point where the central pillar meets the grass slope. [2]

Unfortunately, the workers misunderstand the instructions and build the central pillar from the apex to the floor of the greenhouse such that the pillar is perpendicular to the grass slope.

(iv) Find the length of this central pillar and use this length to find the volume enclosed by the greenhouse *OABC*. [4]

[The volume of a tetrahedron is given by $\frac{1}{3}\times(\text{base area})\times(\text{height})$.]

Another straight supporting beam is to be constructed from point *M*, the midpoint of *OB*, to a point *X* on *OC* such that *XM* is perpendicular to *OB*.

(v) Find the coordinates of *X*, correct to 1 decimal place. [3]

	Suggested solution
(i)	$\overrightarrow{OA}\times\overrightarrow{OB}=\begin{pmatrix}100\\100\\10\end{pmatrix}\times\begin{pmatrix}160\\20\\8\end{pmatrix}=\begin{pmatrix}600\\800\\-14000\end{pmatrix}=200\begin{pmatrix}3\\4\\-70\end{pmatrix}$ Since the origin lies on the plane *OAB*. Plane *OAB*: $\mathbf{r}\cdot\begin{pmatrix}3\\4\\-70\end{pmatrix}=0$ $3x+4y-70z=0$ (shown)
(ii)	Angle of inclination $=\cos^{-1}\dfrac{\left\|\begin{pmatrix}3\\4\\-70\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}3\\4\\-70\end{pmatrix}\right\|\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|}=\cos^{-1}\dfrac{70}{\sqrt{4925}}=4.1^\circ$ (to 1 d.p.)
(iii)	**Method I**: Let F be a point directly below C. Hence coordinate of F is $(80,59,f)$. Since *F* is on plane *OAB*, $\begin{pmatrix}80\\59\\f\end{pmatrix}\cdot\begin{pmatrix}3\\4\\-70\end{pmatrix}=0\Rightarrow 476-70f=0\Rightarrow f=\dfrac{34}{5}=6.8$ Hence the coordinates of the point *F* is $(80,59,\dfrac{34}{5})$ **Method II:** Let the line from apex parallel to **k** be L_c. $L_c:\ \mathbf{r}=\begin{pmatrix}80\\59\\20\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\1\end{pmatrix}\quad\lambda\in\mathbb{R}$ At point of intersection with plane *OAB*, $\begin{pmatrix}80\\59\\20+\lambda\end{pmatrix}\cdot\begin{pmatrix}3\\4\\-70\end{pmatrix}=0\Rightarrow -924-70\lambda=0\Rightarrow\lambda=-\dfrac{66}{5}$ Hence the coordinates of the point *F* is $(80,59,\dfrac{34}{5})$

(iv) Perpendicular distance from C to the plane OAB

$$= \frac{\left| \overrightarrow{OC} \cdot \begin{pmatrix} 3 \\ 4 \\ -70 \end{pmatrix} \right|}{\left| \begin{pmatrix} 3 \\ 4 \\ -70 \end{pmatrix} \right|} = \frac{\left| \begin{pmatrix} 80 \\ 59 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -70 \end{pmatrix} \right|}{\left| \begin{pmatrix} 3 \\ 4 \\ -70 \end{pmatrix} \right|}$$

$$= \frac{924}{\sqrt{4925}} = \frac{924}{5\sqrt{197}} = 13.16645 \approx 13.2 \text{ m (to 3.s.f.)}$$

Volume enclosed by the greenhouse

$$= \frac{1}{3}(\text{Area of } \Delta OAB)(\text{Perpendicular distance from } C \text{ to plane } OAB)$$

$$= \frac{1}{3}\left(\frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right|\right)\left(\frac{924}{\sqrt{4925}}\right)$$

$$= \frac{1}{6}\left|200\begin{pmatrix} 3 \\ 4 \\ -70 \end{pmatrix}\right|\left(\frac{924}{\sqrt{4925}}\right)$$

$$= 30800 \text{ m}^3$$

(v) Since X lies on OC, $\overrightarrow{OX} = t\begin{pmatrix} 80 \\ 59 \\ 20 \end{pmatrix}$, for some $t \in \mathbb{R}$.

Since XM is perpendicular to OB, $\overrightarrow{XM} \cdot \overrightarrow{OB} = 0$

$$\left(\overrightarrow{OM} - \overrightarrow{OX}\right) \cdot \overrightarrow{OB} = 0$$

$$\left(\begin{pmatrix} 80 \\ 10 \\ 4 \end{pmatrix} - t\begin{pmatrix} 80 \\ 59 \\ 20 \end{pmatrix}\right) \cdot \begin{pmatrix} 160 \\ 20 \\ 8 \end{pmatrix} = 0$$

$$13032 - 14140t = 0$$

$$t = \frac{3258}{3535}$$

Therefore, $\overrightarrow{OX} = \frac{3258}{3535}\begin{pmatrix} 80 \\ 59 \\ 20 \end{pmatrix}$.

Coordinates of X are $(73.7, 54.4, 18.4)$.

12 In a model making competition, each contestant is given a square cardboard of side n cm to make a model. John takes part in the Pyramid Category and he is given a $n \times n$ cardboard to make a pyramid with the maximum volume. John recalled his calculus during his schooling days and came out with a design as shown in the diagrams below.

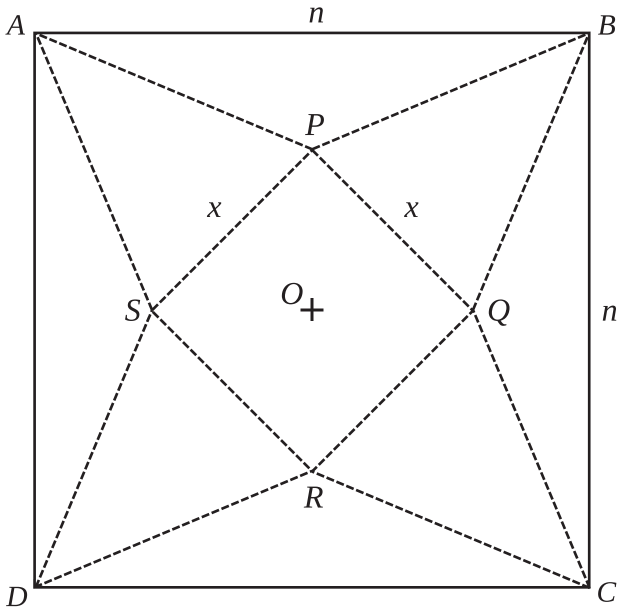


Figure 1

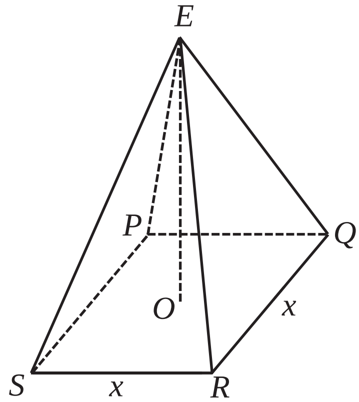


Figure 2

Figure 1 shows a square piece of cardboard $ABCD$ of side n cm, where n is a positive constant. Four triangles (ABP, BCQ, CDR, DAS) are removed from each side of $ABCD$. The remaining shape is now folded along PQ, QR, RS and SP to form a pyramid (as shown in Figure 2) with a square base $PQRS$ of side x cm. The point O is the centre of both $ABCD$ and $PQRS$. The vertex E is the point where A, B, C and D meet. Let V be the volume of the pyramid.

(i) Show that $OE^2 = \frac{1}{2}\left(n^2 - \sqrt{2}nx\right)$. [3]

(ii) Show that $V^2 = \frac{x^4}{18}\left(n^2 - \sqrt{2}nx\right)$. Hence, by using differentiation, find in terms of n, the exact value of x which John uses to obtain a maximum value of V. [7]

(iii) John's pyramid has a volume that is greater than 45 cm^3. Find the smallest $n \times n$ cardboard given, where $n \in \mathbb{Z}^+$. [2]

[The volume of a pyramid is given by $\frac{1}{3} \times (\text{base area}) \times (\text{height})$.]

Suggested solution

(i) <u>Method I</u>

$BD=\sqrt{n^2+n^2}=\sqrt{2}\,n$

Let M be the midpoint of SR, $\therefore DM=EM=\dfrac{\sqrt{2}n-x}{2}$

$EM^2=OE^2+OM^2$

$OE^2=EM^2-OM^2$

$$=\left(\frac{\sqrt{2}n-x}{2}\right)^2-\left(\frac{x}{2}\right)^2$$

$$=\frac{1}{4}\left(2n^2-2\sqrt{2}nx+x^2-x^2\right)$$

$$=\frac{1}{4}\left(2n^2-2\sqrt{2}nx\right)$$

$$=\frac{1}{2}\left(n^2-\sqrt{2}nx\right)\quad(\text{shown})$$

<u>Method II (Not recommended)</u>

$PR=\sqrt{x^2+x^2}=\sqrt{2}\,x$

Let M be the midpoint of DC, $\therefore RM=\dfrac{n-\sqrt{2}\,x}{2}$

$$DR^2=DM^2+RM^2=\left(\frac{n}{2}\right)^2+\left(\frac{n-\sqrt{2}\,x}{2}\right)^2=ER^2$$

$ER^2=OE^2+OR^2$

$OE^2=ER^2-OR^2$

$$=\left[\left(\frac{n}{2}\right)^2+\left(\frac{n-\sqrt{2}\,x}{2}\right)^2\right]-\left(\frac{\sqrt{2}\,x}{2}\right)^2$$

$$=\frac{n^2}{4}+\frac{n^2-2\sqrt{2}\,nx+2x^2}{4}-\frac{2x^2}{4}$$

$$=\frac{1}{4}\left(2n^2-2\sqrt{2}\,nx\right)$$

$$=\frac{1}{2}\left(n^2-\sqrt{2}\,nx\right)\quad(\text{shown})$$

(ii)

$$V=\frac{1}{3}(\text{base area})(\text{height, }OE)$$

$$=\frac{1}{3}x^2\sqrt{\frac{1}{2}\left(n^2-\sqrt{2}nx\right)}$$

$$\Rightarrow V^2=\frac{x^4}{18}\left(n^2-\sqrt{2}nx\right)\quad(\text{shown})$$

Method I (Implicit differentiation and 2nd derivative test)

$$V^2 = \frac{x^4}{18}\left(n^2 - \sqrt{2}nx\right) = \frac{n^2}{18}x^4 - \frac{\sqrt{2}n}{18}x^5$$

Diff wrt x,

$$2V\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{n^2}{18}\left(4x^3\right) - \frac{\sqrt{2}n}{18}\left(5x^4\right)$$
$$= \frac{2n^2}{9}x^3 - \frac{5\sqrt{2}n}{18}x^4$$

When $\frac{\mathrm{d}V}{\mathrm{d}x} = 0 \Rightarrow \frac{2n^2}{9}x^3 - \frac{5\sqrt{2}n}{18}x^4 = 0$

$$\Rightarrow \frac{nx^3}{18}\left(4n - 5\sqrt{2}x\right) = 0$$

$$\Rightarrow x = \frac{4n}{5\sqrt{2}} \quad \text{or} \quad x = 0 \quad \left(\text{rej. } \because x > 0\right)$$
$$= \frac{2\sqrt{2}}{5}n$$

Consider: $\quad 2V\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{2n^2}{9}x^3 - \frac{5\sqrt{2}n}{18}x^4$

Diff wrt x,

$$2\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right) + 2V\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \frac{2n^2}{9}\left(3x^2\right) - \frac{5\sqrt{2}n}{18}\left(4x^3\right)$$
$$2\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 + 2V\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \frac{2n^2}{3}x^2 - \frac{10\sqrt{2}n}{9}x^3$$

Substituting $x = \frac{2\sqrt{2}n}{5}$, $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$,

$$2(0)^2 + 2V\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \frac{2n^2}{3}\left(\frac{2\sqrt{2}n}{5}\right)^2 - \frac{10\sqrt{2}n}{9}\left(\frac{2\sqrt{2}n}{5}\right)^3$$
$$2V\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \frac{2n^2}{3}\left(\frac{8n^2}{25}\right) - \frac{10\sqrt{2}n}{9}\left(\frac{16\sqrt{2}n^3}{125}\right)$$
$$= \frac{16n^4}{75} - \frac{64n^4}{225}$$
$$= -\frac{16n^4}{225}$$
$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -\frac{8n^4}{225V} < 0 \quad \left(\text{since } V, n > 0\right)$$

$\therefore V$ is maximum when $x = \frac{2\sqrt{2}}{5}n$.

Method II (Making V the subject, differentiating and using 1st derivative test) – Not recommended

$$V=\frac{1}{3}x^2\sqrt{\frac{1}{2}\left(n^2-\sqrt{2}nx\right)}$$

$$=\frac{1}{3\sqrt{2}}x^2\left(n^2-\sqrt{2}nx\right)^{\frac{1}{2}}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x}=\frac{1}{3\sqrt{2}}\left\{2x\left(n^2-\sqrt{2}nx\right)^{\frac{1}{2}}+x^2\left[\frac{1}{2}\left(n^2-\sqrt{2}nx\right)^{-\frac{1}{2}}\left(-\sqrt{2}n\right)\right]\right\}$$

$$=\frac{1}{3\sqrt{2}}\left\{2x\left(n^2-\sqrt{2}nx\right)^{\frac{1}{2}}+x^2\left[\frac{-\sqrt{2}n}{2\left(n^2-\sqrt{2}nx\right)^{\frac{1}{2}}}\right]\right\}$$

$$=\frac{x}{6\sqrt{2}}\left\{\left[\frac{4\left(n^2-\sqrt{2}nx\right)}{\left(n^2-\sqrt{2}nx\right)^{\frac{1}{2}}}\right]+x\left[\frac{-\sqrt{2}n}{\left(n^2-\sqrt{2}nx\right)^{\frac{1}{2}}}\right]\right\}$$

$$=\frac{x}{6\sqrt{2}}\left[\frac{4n^2-4\sqrt{2}nx-\sqrt{2}nx}{\left(n^2-\sqrt{2}nx\right)^{\frac{1}{2}}}\right]$$

$$=\frac{nx}{6\sqrt{2}}\left[\frac{4n-5\sqrt{2}x}{\left(n^2-\sqrt{2}nx\right)^{\frac{1}{2}}}\right]$$

When $\frac{\mathrm{d}V}{\mathrm{d}x}=0\Rightarrow x\left(4n-5\sqrt{2}x\right)=0$

$$\Rightarrow x=\frac{4n}{5\sqrt{2}}\quad\text{or}\quad x=0\quad\left(\text{rej. }\because x>0\right)$$

$$=\frac{2\sqrt{2}}{5}n$$

x	$\left(\frac{2\sqrt{2}}{5}n\right)^-$	$\frac{2\sqrt{2}}{5}n$	$\left(\frac{2\sqrt{2}}{5}n\right)^+$
Explanation	$4n-5\sqrt{2}\,x>0$ $x>0$	$4n-5\sqrt{2}\,x=0$	$4n-5\sqrt{2}\,x<0$ $x>0$
$\frac{\mathrm{d}V}{\mathrm{d}x}$	+ve	0	–ve
Slope	/	–	\

$\therefore V$ is maximum when $x=\frac{2\sqrt{2}}{5}n$.

(iii) **<u>Method I (Inequalities without GC table method)</u>**

V is maximum when $x=\dfrac{2\sqrt{2}}{5}n, V>45 \Rightarrow V^2>45^2$

From part **(ii)**, $V^2=\dfrac{x^4}{18}\left(n^2-\sqrt{2}nx\right)=\dfrac{n^2}{18}x^4-\dfrac{\sqrt{2}n}{18}x^5$

$$\frac{n^2}{18}\left(\frac{2\sqrt{2}}{5}n\right)^4-\frac{\sqrt{2}n}{18}\left(\frac{2\sqrt{2}}{5}n\right)^5>45^2$$

$$\frac{32}{5625}n^6-\frac{128}{28125}n^6>2025$$

$$\frac{32}{28125}n^6>2025$$

$$n>11.008 \quad (5\text{s.f.})$$

Least integer n is 12. Smallest cardboard given is $12\text{cm} \times 12\text{cm}$.

<u>Method II (Inequalities using GC table method) - Recommended</u>

V is maximum when $x=\dfrac{2\sqrt{2}}{5}n, V>45$

From part **(ii)**, $V^2=\dfrac{x^4}{18}\left(n^2-\sqrt{2}nx\right) \Rightarrow V=\sqrt{\dfrac{x^4}{18}\left(n^2-\sqrt{2}nx\right)}$

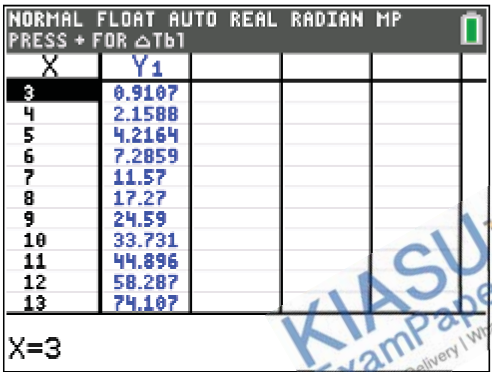
$$\sqrt{\frac{n^2}{18}\left(\frac{2\sqrt{2}}{5}n\right)^4-\frac{\sqrt{2}n}{18}\left(\frac{2\sqrt{2}}{5}n\right)^5}>45$$

Using GC,

When $n=11, V=44.9<45$

When $n=12, V=58.3>45$

Smallest cardboard given is $12\text{cm} \times 12\text{cm}$.


NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl

X	Y1
3	0.9107
4	2.1588
5	4.2164
6	7.2859
7	11.57
8	17.27
9	24.59
10	33.731
11	44.896
12	58.287
13	74.107

X=3