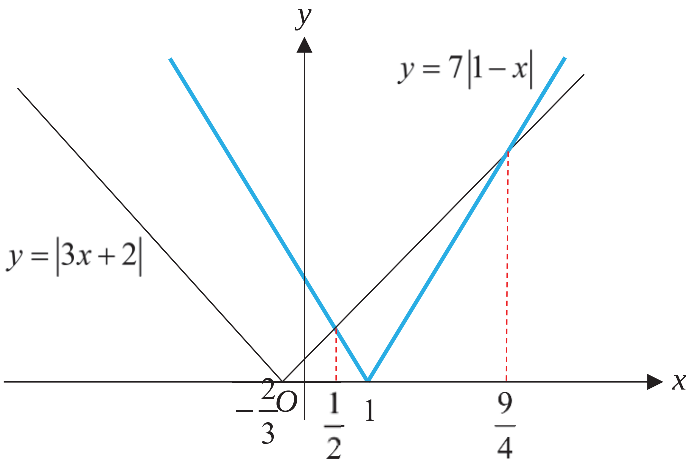
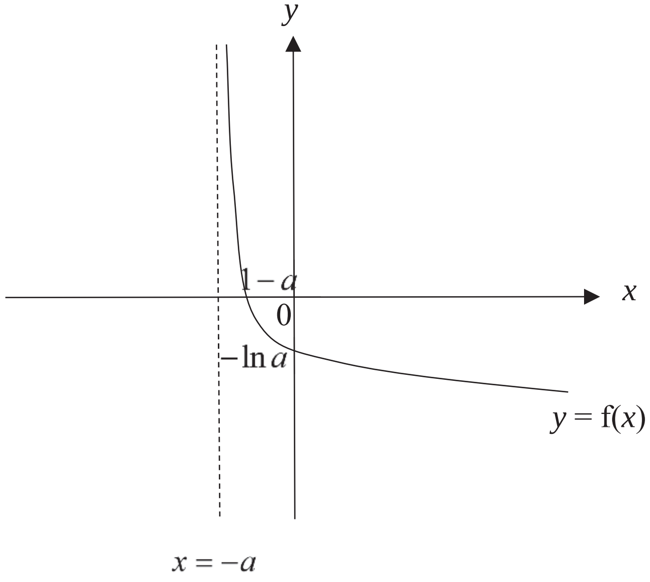
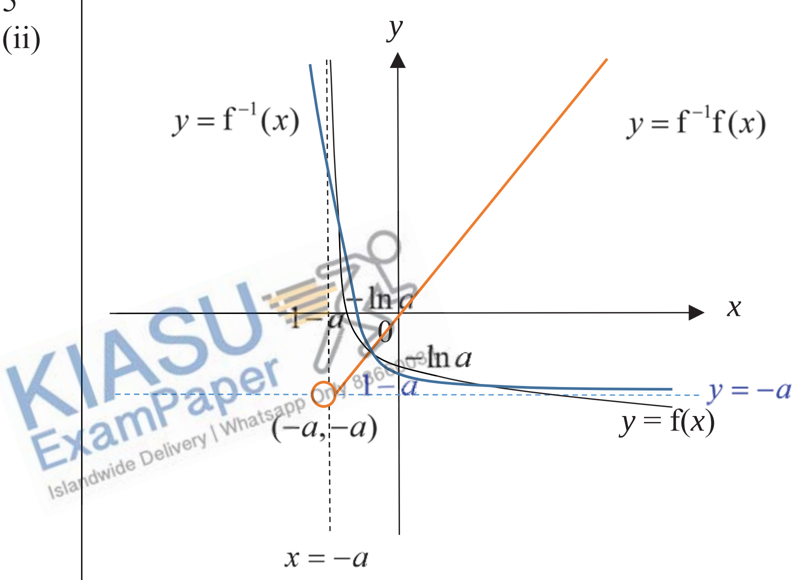
St Andrew's Junior College
2019 Final Examinations
H2 Mathematics (9758) Solutions

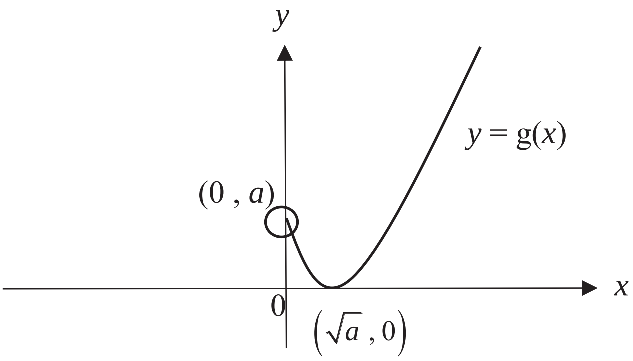
1	 Using G.C., the x-coordinates of the intersections are $\frac{1}{2}$ and $\frac{9}{4}$. From the diagram, $\|3x+2\| \geq 7\|1-x\|$ when $\frac{1}{2} \leq x \leq \frac{9}{4}$.

2	$x^{\tan^{-1}x} = y^{x+1}$ Applying ln on both sides, $\ln\left(x^{\tan^{-1}x}\right) = \ln\left(y^{x+1}\right)$ $\left(\tan^{-1}x\right)\ln x = (x+1)\ln y ------(*)$ Differentiate (*) with respect to *x*, $\frac{\ln x}{1+x^2} + \frac{\tan^{-1}x}{x} = \ln y + (x+1)\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$ $x = 1, 1^{\tan^{-1}1} = y^2$ $\Rightarrow y = \pm 1$ (Reject $y = -1$ since $y>0$) $0 + \tan^{-1}1 = 0 + 2\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\pi}{8}$ The gradient of the curve at $x = 1$ is $\frac{\pi}{8}$. (exact answer is required)

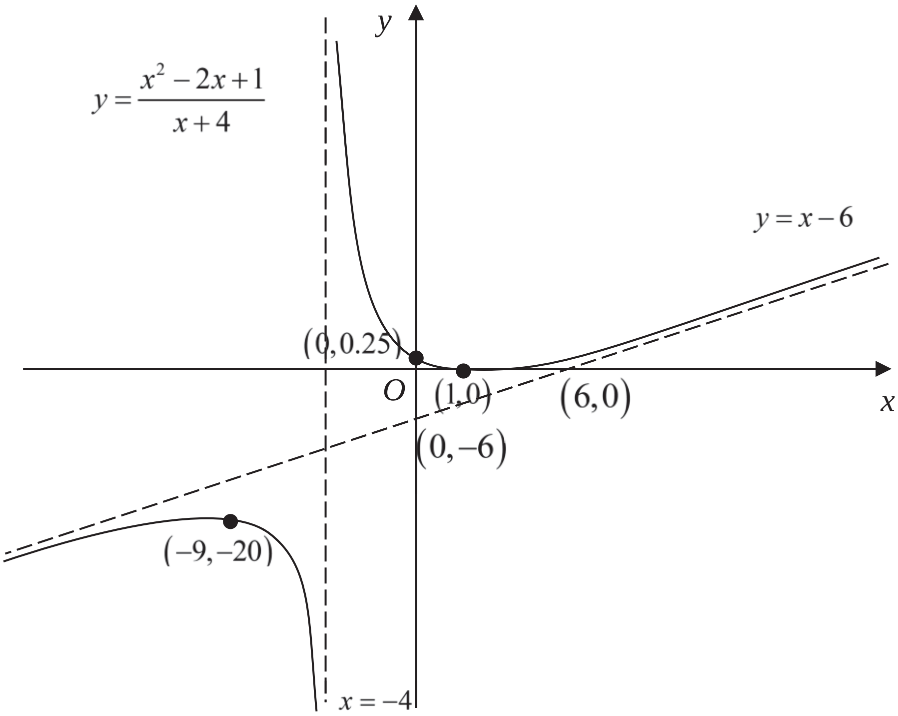
3(i)	Let u_n and S_n be the nth term and sum of the first n terms of the sequence respectively. Given $S_n = \dfrac{3n^2 - n}{2}$, then $S_{n-1} = \dfrac{3(n-1)^2 - (n-1)}{2}$. $u_n = S_n - S_{n-1}$ $= \dfrac{3n^2 - n}{2} - \left[\dfrac{3(n-1)^2 - (n-1)}{2}\right]$ $= \dfrac{3n^2 - n - 3(n-1)^2 + (n-1)}{2}$ $= \dfrac{3n^2 - n - 3(n^2 - 2n + 1) + (n-1)}{2}$ $= \dfrac{3n^2 - n - 3n^2 + 6n - 3 + n - 1}{2}$ $= \dfrac{6n-4}{2}$ $= 3n - 2$ $u_n - u_{n-1}$ $= (3n-2) - [3(n-1) - 2]$ $= (3n-2) - (3n-3-2)$ $= 3n - 2 - 3n + 3 + 2$ $= 3$ Since $u_n - u_{n-1}$=3 is a **constant independent of *n***, the sequence is an arithmetic progression.

3 (ii)	$\sum_{r=1}^{n}\left(\frac{2^{u_r}}{u_{r+2}-u_r}+r\right)$ $=\sum_{r=1}^{n}\left[\frac{2^{3r-2}}{6}+r\right]$ $=\sum_{r=1}^{n}\left(\frac{2^{3r}}{2^2\times 6}\right)+\sum_{r=1}^{n}r$ $=\sum_{r=1}^{n}\frac{8^r}{24}+\frac{n}{2}(1+n)$ $=\frac{1}{24}\sum_{r=1}^{n}8^r+\frac{n(n+1)}{2}$ $=\frac{1}{24}\left[\frac{8(8^n-1)}{8-1}\right]+\frac{n(n+1)}{2}$ $=\frac{1}{21}(8^n-1)+\frac{n(n+1)}{2}$
4 (i)	$(2\mathbf{a}+\mathbf{b})\bullet(3\mathbf{a}-5\mathbf{b})$ $=6\mathbf{a}\bullet\mathbf{a}-5\mathbf{b}\bullet\mathbf{b}+3\mathbf{b}\bullet\mathbf{a}-10\mathbf{a}\bullet\mathbf{b}$ $=6\lvert\mathbf{a}\rvert^2-5\lvert\mathbf{b}\rvert^2-7\mathbf{a}\bullet\mathbf{b}$ $=6(4^2)-5(1^2)-7(0)$ $=91$
4 (ii)	By Ratio theorem, $\mathbf{c}=\frac{\mathbf{a}+3\mathbf{b}}{4}$.
4 (iii)	$\lvert\mathbf{b}\times\mathbf{c}\rvert$ is the area of the parallelogram with *OB* and *OC* as two adjacent sides. Or: $\lvert\mathbf{b}\times\mathbf{c}\rvert$ is the shortest distance from the point *C* to the line passing through *O* and *B*.
	$\lvert\mathbf{b}\times\mathbf{c}\rvert=\left\lvert\mathbf{b}\times\left(\frac{\mathbf{a}+3\mathbf{b}}{4}\right)\right\rvert$ $=\frac{1}{4}\lvert(\mathbf{b}\times\mathbf{a})+3(\mathbf{b}\times\mathbf{b})\rvert$ $=\frac{1}{4}\lvert(\mathbf{b}\times\mathbf{a})+3(\mathbf{0})\rvert$ $=\frac{1}{4}\lvert\mathbf{b}\times\mathbf{a}\rvert$ $=\frac{1}{4}\lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert\sin 90°$ $=\frac{1}{4}(1)(4)(1)=1$

5(i)	 Since any/every horizontal line $y = k,\ k \in \mathbb{R}$ intersects the graph of $y = \mathrm{f}(x)$ at exactly one point, hence f is one-to-one and f^{-1} exists.
5 (ii)	

5 (iii)	 $D_g=(0,\infty)\xrightarrow{g}R_g=[0,\infty)\xrightarrow{f}R_{fg}=(-\infty,-\ln a]$
5 (iv)	$f^{-1}g(x)=e^{-3}-a$ $ff^{-1}g(x)=f(e^{-3}-a)$ $g(x)=-\ln(e^{-3}-a+a)$ $g(x)=\ln e^{3}=3$ $(x-\sqrt{a})^2=3$ $x=\sqrt{a}\pm\sqrt{3}$ $1<a\le 3$ $1<\sqrt{a}\le\sqrt{3}$ $\therefore \sqrt{a}-\sqrt{3}\le 0$ Since $x\in D_{f^{-1}g}=(0,\infty)$, $x=\sqrt{a}+\sqrt{3}$

6 (i)	Since $x = -4$ is an asymptote, $\Rightarrow \quad b = 4$ $y = \dfrac{x^2 + ax + 1}{x+4} = x + (a-4) + \dfrac{1-4(a-4)}{x+4}$ Oblique asymptote: $y = x + a - 4 = x - 6$ Comparing, $a - 4 = -6$ $a = -2$ Hence, $y = \dfrac{x^2 - 2x + 1}{x+4} = x - 6 + \dfrac{25}{x+4}$
	Alternatively (for the value of *a*) Since $y = x - 6$ is an asymptote, $y = x - 6 + \dfrac{A}{x+4} = \dfrac{(x-6)(x+4) + A}{x+4} = \dfrac{x^2 - 2x - 24 + A}{x+4}$ Comparing $\dfrac{x^2 - 2x - 24 + A}{x+4}$ with $\dfrac{x^2 + ax + 1}{x+4}$, We have $a = -2$ and $-24 + A = 1$ $A = 25$ Thus $y = \dfrac{x^2 - 2x + 1}{x+4} = x - 6 + \dfrac{25}{x+4}$

6 (ii)	
6 (iii)	Let *C*': Reflection about the *x* – axis *B*': Scaling parallel to the *x* – axis by a scale factor of $\frac{1}{2}$ *A*': Translation of 3 units in the positive *x* direction

$$y=\frac{x^2-2x+1}{x+4}$$

$\downarrow$ C': replace y by $-y$

$$-y=\frac{x^2-2x+1}{x+4}$$

$\downarrow$ B': replace x by $\frac{x}{\frac{1}{2}}=2x$

$$-y=\frac{(2x)^2-2(2x)+1}{(2x)+4}$$

$$-y=\frac{4x^2-4x+1}{2x+4}$$

$\downarrow$ A': replace x by $x-3$

$$-y=\frac{4(x-3)^2-4(x-3)+1}{2(x-3)+4}$$

Therefore equation of C_2 is

$$y=-\frac{4(x-3)^2-4x+13}{2x-2}=\frac{-4x^2+28x-49}{x-22}$$

Q7

nth month	Amount in the account at the start of the month	Amount in the account at the end of the month
1	20000	$20000(1.0035)$
2	$20000(1.0035)-1500$	$\left[20000(1.0035)-1500\right](1.0035)$ $=20000(1.0035)^2-1500(1.0035)$
3	$20000(1.0035)^2-1500(1.0035)-1500$	$20000(1.0035)^3-1500(1.0035)^2$ $-1500(1.0035)$
4	$20000(1.0035)^3-1500(1.0035)^2$ $-1500(1.0035)-1500$	$20000(1.0035)^4-1500(1.0035)^3$ $-1500(1.0035)^2-1500(1.0035)$
…		
n	$20000(1.0035)^{n-1}-1500(1.0035)^{n-2}$ $-1500(1.0035)^{n-3}-\ldots$ $-1500(1.0035)-1500$	$20000(1.0035)^n-1500(1.0035)^{n-1}$ $-1500(1.0035)^{n-2}$ $-1500(1.0035)^{n-3}-\ldots$ $-1500(1.0035)$

7 (Cont'd)	At the end of the nth month, $20000(1.0035)^n-1500(1.0035)^{n-1}-1500(1.0035)^{n-2}-\ldots-1500(1.0035)$ $=20000(1.0035)^n-1500\left[(1.0035)^{n-1}+\ldots+1.0035\right]$ $=20000(1.0035)^n-1500\left[1.0035+\ldots+(1.0035)^{n-1}\right]$ $=20000(1.0035)^n-1500(1.0035)\left[\frac{(1.0035)^{n-1}-1}{1.0035-1}\right]$ $=20000(1.0035)^n-(430071.4286)\left[(1.0035)^{n-1}-1\right]$

If Clarence cannot draw another \$1500 in the following month,

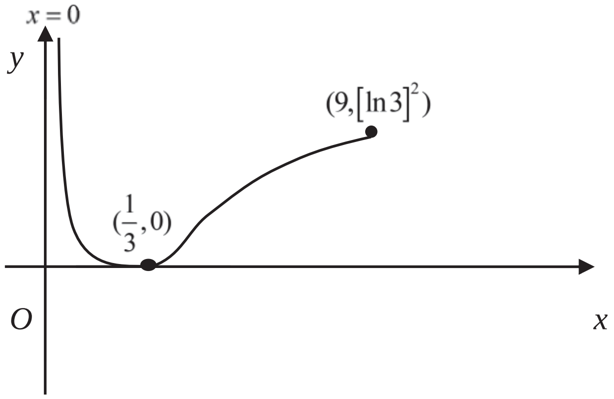
$$20000(1.0035)^n - (430071.4286)\left[(1.0035)^{n-1} - 1\right] < 1500$$

Using GC,

n	$20000(1.0035)^n - (430071.4286)\left[(1.0035)^{n-1} - 1\right]$
13	2514.6 > 1500
14	1018.1 < 1500
15	-483.6 < 1500

Hence, Clarence's last draw of \$ 1500 is on the 14^{th} month, and hence draw a **maximum of 13 months** from the bank account.

The maximum number of draws = 13.

8(i)	 Asymptote: $t=0, x=0$ Intercept: $y=0, t=1 \Rightarrow x=\frac{1}{3}$
(ii)	$\frac{dx}{dt}=\frac{3t^2}{3}=t^2, \quad \frac{dy}{dt}=\frac{2\ln(t)}{t}$ $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{1}{\frac{dx}{dt}}$ $=\frac{2\ln(t)}{t}\times\frac{1}{t^2}$. $=\frac{2\ln(t)}{t^3}$ At $t=p, \ \frac{dy}{dx}=\frac{2\ln(p)}{p^3}$ Equation of the tangent of C at point p : $y-\left[\ln(p)\right]^2=\frac{2\ln(p)}{p^3}\left(x-\frac{p^3}{3}\right)$ $y-[\ln p]^2=\left(\frac{2}{p^3}\ln p\right)x-\frac{2}{3}\ln p$ $y=\left(\frac{2}{p^3}\ln p\right)x+[\ln p]^2-\frac{2}{3}\ln p$
(iii)	Equation of the tangent of C at point $t=p$: $y=\left(\frac{2}{p^3}\ln p\right)x+[\ln p]^2-\frac{2}{3}\ln p$ At $p=e$,

	$y=\left(\frac{2}{e^3}\ln e\right)x+\left[\ln e\right]^2-\frac{2}{3}\ln e$ $y=\frac{2}{e^3}x+\frac{1}{3}$ When the tangent cuts the axis at $x-$axis, $y=0$ $x=-\frac{e^3}{6}$ When the tangent cuts the axis at $y-$axis, $x=0\Rightarrow y=\frac{1}{3}$ The coordinates of Q is $\left(-\frac{e^3}{6},0\right)$. The coordinates of R is $(0,\frac{1}{3})$.
(iv)	Area of triangle $OQR=\frac{1}{2}\left(\frac{e^3}{6}\right)\left(\frac{1}{3}\right)$ $=\frac{e^3}{36}$ units^2

9(i)	$\sin(r+\frac{1}{2})\theta-\sin(r-\frac{1}{2})\theta$ $=2\cos\frac{1}{2}\left[(r+\frac{1}{2})\theta+(r-\frac{1}{2})\theta\right]\sin\frac{1}{2}\left[(r+\frac{1}{2})\theta-(r-\frac{1}{2})\theta\right]$ $=2\cos\frac{1}{2}[2r\theta]\sin\frac{1}{2}[\theta]$ $=2\cos r\theta\sin\frac{1}{2}\theta$ (shown)
9 (ii)	$\sum_{r=1}^{n}\cos r\theta$ $=\frac{1}{2\sin\frac{\theta}{2}}\sum_{r=1}^{n}\left[\sin(r+\frac{1}{2})\theta-\sin(r-\frac{1}{2})\theta\right]$ $=\frac{1}{2\sin\frac{\theta}{2}}[\sin\frac{3}{2}\theta-\sin\frac{1}{2}\theta$ $+\sin\frac{5}{2}\theta-\sin\frac{3}{2}\theta$ $+\sin\frac{7}{2}\theta-\sin\frac{5}{2}\theta$ $+\ldots.$ $+\sin\left(n-\frac{3}{2}\right)\theta-\sin\left(n-\frac{5}{2}\right)\theta$ $+\sin\left(n-\frac{1}{2}\right)\theta-\sin\left(n-\frac{3}{2}\right)\theta$ $+\sin\left(n+\frac{1}{2}\right)\theta-\sin\left(n-\frac{1}{2}\right)\theta]$ $=\frac{\sin\left(n+\frac{1}{2}\right)\theta-\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$ $=\frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}-\frac{1}{2}$ $p=n+\frac{1}{2};\ q=\frac{1}{2}$

(iii)	$\cos 2\theta+\cos 4\theta+\cos 6\theta+...+\cos 2n\theta$ $=\sum_{r=1}^{n}\cos 2r\theta$ $=\dfrac{\sin\left(n+\frac{1}{2}\right)2\theta}{2\sin\frac{2\theta}{2}}-\dfrac{1}{2}$ (Replace θ with 2θ in (ii)) $=\dfrac{\sin(2n+1)\theta}{2\sin\theta}-\dfrac{1}{2}$ $\cos\theta+\cos 3\theta+\cos 5\theta+...+\cos 97\theta+\cos 99\theta$ $=\cos\theta+\cos 2\theta+\cos 3\theta+...+\cos 100\theta$ $-(\cos 2\theta+\cos 4\theta+\cos 6\theta+...+\cos 100\theta)$ $=\sum_{r=1}^{100}\cos r\theta-\sum_{r=1}^{50}\cos 2r\theta$ $=\dfrac{\sin\left(100+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}-\dfrac{1}{2}-\left[\dfrac{\sin(100+1)\theta}{2\sin\theta}-\dfrac{1}{2}\right]$ $=\dfrac{\sin\left(\frac{201}{2}\right)\theta}{2\sin\frac{\theta}{2}}-\dfrac{\sin(101)\theta}{2\sin\theta}$ $=\dfrac{\sin\left(\frac{201}{2}\theta\right)\sin\theta-\sin(101\theta)\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}\sin\theta}$

10 (i)	$\prod_1 : \mathbf{r} \bullet \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 22$ Since F is the foot of perpendicular of P on $\prod_1$, $l_{PF} : \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$. Since F lies on the line, $\overrightarrow{OF} = \begin{pmatrix} 3-\lambda \\ -2+2\lambda \\ 1+2\lambda \end{pmatrix}$, for some $\lambda \in \mathbb{R}$ Since F lies on $\prod_1$, $\begin{pmatrix} 3-\lambda \\ -2+2\lambda \\ 1+2\lambda \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 22$ $-(3-\lambda)+2(-2+2\lambda)+2(1+2\lambda)=22$ $9\lambda = 27$ $\lambda = 3$ $\overrightarrow{OF} = \begin{pmatrix} 3-3 \\ -2+6 \\ 1+6 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 7 \end{pmatrix}$
(ii)	Direction vectors to plane $\prod_2$ are $\begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix}$

	To find the normal of the plane $\prod_2$: $\begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix}$ $= \begin{pmatrix} 6-(-2) \\ -(-6-10) \\ -3-(-15) \end{pmatrix}$ $= \begin{pmatrix} 8 \\ 16 \\ 12 \end{pmatrix}$ $= 4\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$ $\sin\theta = \dfrac{\left|\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}\right|}{\sqrt{(-1)^2+(2)^2+(2)^2}\sqrt{(2)^2+(4)^2+(3)^2}}$ $= \dfrac{4}{\sqrt{29}}$ $= \dfrac{4\sqrt{29}}{29}$

(iii)	Let N be the midpoint of FG. By ratio theorem, $\overrightarrow{ON} = \dfrac{\overrightarrow{OF} + \overrightarrow{OG}}{2}$ $= \dfrac{1}{2}\left[\begin{pmatrix} 0 \\ 4 \\ 7 \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \dfrac{1}{2}\begin{pmatrix} x \\ 4+y \\ 7+z \end{pmatrix}$
(iv)	$\Pi_2 : \mathbf{r}\bullet\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}\bullet\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = 6-8+3=1$ Since N lies on Π_2, $\overrightarrow{ON}\bullet\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = 1$ $\therefore \dfrac{1}{2}\begin{pmatrix} x \\ 4+y \\ 7+z \end{pmatrix}\bullet\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = 1$ $2x+4(4+y)+3(7+z)=2$ $2x+16+4y+21+3z=2$ $2x+4y+3z=2-16-21=-35$ $\therefore$ The cartesian equation describing the set of all points which G may take is $2x+4y+3z=-35$. The set of all points which G may take is a plane parallel to Π_2.

11 (i)	Let the radius of the cup be r m and the height of the cup be h m. $2\pi r = a\theta$ $r = \frac{a\theta}{2\pi}$ By Pythagoras Theorem, $h = \sqrt{a^2 - r^2}$ Volume of the cup, $V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi\left(\frac{a\theta}{2\pi}\right)^2 \sqrt{a^2 - \left(\frac{a\theta}{2\pi}\right)^2}$ $= \frac{a^2\theta^2}{12\pi}\sqrt{\frac{a^2 4\pi^2 - a^2\theta^2}{(2\pi)^2}}$ $= \frac{a^2\theta^2}{12\pi}\frac{\sqrt{a^2(4\pi^2-\theta^2)}}{2\pi}$ $= \frac{a^3}{24\pi^2}\theta^2\sqrt{4\pi^2-\theta^2}\ m^3$
(ii)	$\frac{dV}{d\theta} = \frac{a^3}{24\pi^2}\left[2\theta\sqrt{4\pi^2-\theta^2} + \theta^2\left(\frac{-2\theta}{2\sqrt{4\pi^2-\theta^2}}\right)\right]$ $= \frac{a^3}{24\pi^2}\left[\frac{2\theta(4\pi^2-\theta^2)-\theta^3}{\sqrt{4\pi^2-\theta^2}}\right]$ $= \frac{a^3}{24\pi^2}\left[\frac{\theta(8\pi^2-3\theta^2)}{\sqrt{4\pi^2-\theta^2}}\right]$ When V is a stationary value, $\frac{dV}{d\theta} = 0$ $\theta(8\pi^2-3\theta^2) = 0$ $\theta = 0$ or $\theta^2 = \frac{8\pi^2}{3}$ (*reject* since θ>0)

	Maximum $V = \dfrac{a^3}{24\pi^2}\left(\dfrac{8\pi^2}{3}\right)\sqrt{4\pi^2 - \dfrac{8\pi^2}{3}}$ $= \dfrac{a^3}{9}\sqrt{\dfrac{4\pi^2}{3}}$ $= \dfrac{a^3}{9}\dfrac{2\pi}{\sqrt{3}}$ $= \dfrac{a^3}{9}\dfrac{2\pi}{\sqrt{3}}\dfrac{\sqrt{3}}{\sqrt{3}}$ $= \dfrac{2\sqrt{3}\pi a^3}{27}\ m^3$
(iii)