Questions from 2019 SAJC Promos

1 Solve the inequality $|3x+2| \geq 7|1-x|$. **[4]**

2 Given that a curve has the equation $x^{\tan^{-1} x} = y^{x+1}$ where $x > 0$ and $y > 0$, using a non-calculator method, find the exact gradient of the curve at the point where $x = 1$. **[5]**

3 removed (not in syllabus)

4 Relative to the origin O, the points A and B have position vectors **a** and **b** respectively. It is given that the magnitude of **a** is 4 and **b** is a unit vector perpendicular to **a**.

(i) Find the value of $(2\mathrm{a}+\mathrm{b})\bullet(3\mathrm{a}-5\mathrm{b})$. **[4]**

(ii) The point C is on AB such that $AC:CB = 3:1$. Write down the position vector of C, **c**, in terms of **a** and **b**. **[1]**

(iii) State the geometrical meaning of $|\mathrm{b}\times\mathrm{c}|$ and find its exact value. **[5]**

5 Functions f and g are defined by

$$\mathrm{f}: x \mapsto -\ln(x+a),\ x \in \mathbb{R},\ x > -a$$

$$\mathrm{g}: x \mapsto (x-\sqrt{a})^2,\ x \in \mathbb{R},\ x > 0$$

where a is a positive constant such that $1 < a \leq 3$.

(i) Sketch the graph of $y = \mathrm{f}(x)$ and show that f has an inverse. **[4]**

(ii) On the same diagram in part (i), sketch the graph of $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{f}^{-1}\mathrm{f}(x)$. **[2]**

(iii) Find the range of the composite function fg. **[2]**

(iv) Without finding $\mathrm{f}^{-1}(x)$, find x given that $\mathrm{f}^{-1}\mathrm{g}(x) = \mathrm{e}^{-3} - a$. **[3]**

6 The curve C_1 has equation $y = \dfrac{x^2+ax+1}{x+b}$, where $x \in \mathbb{R}$, $x \neq -b$ and a and b are constants. The lines $x = -4$ and $y = x-6$ are asymptotes to C_1.

(i) Write down the value of b. Hence, show that $a = -2$. **[3]**

With the values of a and b found in **(i)**,

(ii) sketch C_1, stating the equations of any asymptotes, the coordinates of any turning points and any points of intersection with the axes, **[3]**

(iii) A second curve C_2 undergoes, in succession, the following transformations to get the resulting curve C_1:

A: Translation of 3 units in the negative x direction;
B: Scaling parallel to the x – axis by a scale factor of 2;
C: Reflection about the x – axis.

Find the equation of the curve C_2, showing your workings clearly. **[3]**

7 removed (not in syllabus)

8 The curve C has parametric equations

$$x=\frac{t^3}{3}, \quad y=\left[\ln(t)\right]^2, \quad \text{for } 0<t\le 3.$$

(i) Sketch the graph of C, giving the coordinates of its endpoint(s) and the point(s) where C meets the axes. State also the equation of the vertical asymptote. **[3]**

(ii) Find the equation of the tangent to the curve C at the point $\left(\frac{p^3}{3},\left[\ln(p)\right]^2\right)$, simplifying your answer. **[5]**

(iii) Hence find the exact coordinates of the points Q and R where the tangent to the curve C when $t=\mathrm{e}$ meets the x-axis and y-axis respectively. **[3]**

(iv) Find the area of triangle OQR in exact form. **[2]**

9 removed (not in syllabus)

10 The plane Π_1 is defined by the equation $\mathbf{r}\bullet(-\mathbf{i}+2\mathbf{j}+2\mathbf{k})=22$. A point P has coordinates $(3,-2,1)$.

(i) Find the position vector of the foot of perpendicular, F, from the point P on Π_1. **[3]**

The line l has equation $\frac{x-3}{3}=\frac{y+2}{-3}=\frac{z-1}{2}$. The plane Π_2 contains the line l and is perpendicular to a plane with normal $\begin{pmatrix}5\\-1\\-2\end{pmatrix}$.

(ii) Find $\sin\theta$, where θ is the acute angle between the plane Π_2 and the line PF. **[4]**

(iii) A general point G has coordinates (x,y,z). Find the position vector of N, the midpoint of FG. **[1]**

(iv) Given that point N described in **(iii)** always lies in Π_2, find a cartesian equation that describes the set of points which G may take. Hence, describe the relationship between the set of points G and the plane Π_2. **[4]**

11 [The volume of a cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$ and the arc length of a sector of radius r and angle θ radians is $r\theta$.]

Figure 1 shows a sector AOB of θ radians which is cut from a circular card of fixed radius a metres with centre O. A cup in the shape of an inverted right circular cone with radius r and height h is then formed by joining the two radii, OA and OB, of the sector together, without overlap (as shown in **Figure 2**).

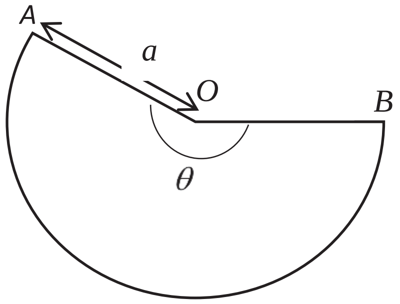

Figure 1

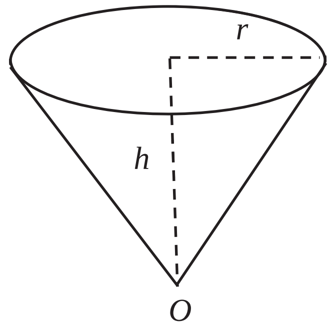

Figure 2

(i) Show that the volume of the cup in **Figure 2**, V cubic metres is given by

$$V = \frac{a^3}{24\pi^2}\theta^2\sqrt{4\pi^2 - \theta^2}\,.$$ **[4]**

(ii) Use differentiation to find, in terms of a, the exact maximum volume of the cup as θ varies. You are not required to justify that the volume of the cup is a maximum. **[5]**

(iii) Hence, sketch the graph showing the volume of the cup, V as the angle of the sector AOB, θ varies. **[3]**