1. **[SAJC 15 Prelims (modified)]**

   (a) Show that
   $$\frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+2} \equiv \frac{r+4}{r(r+1)(r+2)}.$$
   [2]

   (b) Hence find
   $$\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)},$$
   in terms of n. [4]

   (c) Hence find
   $$\sum_{r=2}^{n} \frac{r+3}{r(r^2-1)}.$$
   [3]

2. **[SRJC 15 Prelims (modified)]**

   (a) By considering $\dfrac{1}{n^3} - \dfrac{1}{(n+1)^3}$, find
   $$\sum_{n=1}^{N} \frac{3n^2+3n+1}{n^3(n+1)^3}$$
   in terms of N. [4]

   (b) Hence find
   $$\sum_{n=3}^{N} \frac{3n^2-3n+1}{n^3(n-1)^3}$$
   in terms of N. [3]

   (c) Give a reason why the following sum
   $$\frac{7}{1^3(2^3)} + \frac{19}{2^3(3^3)} + \frac{37}{3^3(4^3)} + \cdots$$
   converges, and state the sum to infinity. [2]

3. **[NJC 15 Prelims (modified)]**

(a) Prove by the method of differences that

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+3)} = \frac{1}{3} - \frac{n+1}{(2n+1)(2n+3)}.$$

[3]

(b) Hence find the exact value of

$$\sum_{r=5}^{\infty} \frac{1}{(2r-1)(2r+3)}.$$

[3]

(c) Hence find

$$\sum_{r=0}^{n} \frac{1}{(2r+1)(2r+5)}$$

in terms of n. [2]

(d) Hence prove that

$$\sum_{r=2}^{n} \frac{1}{(2r-1)^3} < \frac{4}{15}.$$

[2]

4. **[RVHS 15 Prelims (modified)]**

It is given that $u_n = \dfrac{n+1}{n!}$ for $n \in \mathbb{Z}, n \geq 0$.

(a) By considering $u_{r+1} - u_r$, find

$$S_n = \sum_{r=1}^{N} \frac{(r+1)^2 + r}{(r+2)!}.$$

[5]

(b) Give a reason why S_n is convergent and state its sum to infinity. [2]

(c) Find the smallest value of n for which S_n is within 10^{-9} of S_∞. [2]

5. **[JJC 15 Prelims (modified)]**

(a) It is given that $f(r) = 2r^3 + 3r^2 + r$. Show that

$$f(r) - f(r-1) = ar^2,$$

where a is a constant to be determined.

Hence find a formula for $\sum_{r=1}^{n} r^2$, fully factorizing your answer. [4]

(b) It is further given that

$$\sum_{r=1}^{n} (4r^3 + 3r^2 + r) = n(n+1)^3.$$

Find a formula for $\sum_{r=1}^{n} r^3$, fully factorizing your answer.

6. **[CJC 14 Prelims]**

(a) Use the method of differences to show that

$$\sum_{n=2}^{N} \frac{1}{n^2 - 1} = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{N} - \frac{1}{N+1}\right).$$

[4]

(b) Find $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ and show that $\sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}$. [4]

7. **[PJC 13 Prelims (modified)]**

(a) Express

$$\frac{2}{(r+1)(r+3)}$$

in partial fractions and find

$$\sum_{r=1}^{N} \frac{1}{(r+1)(r+3)}.$$

[4]

(b) Hence find

$$\sum_{r=6}^{N+4} \frac{1}{(r-3)(r-1)}.$$

[2]

8. **[VJC 14 Prelims]**

(a) Given that $\dfrac{(n+1)}{n^2(n+2)^2} = a\left(\dfrac{1}{n^2} - \dfrac{1}{(n+2)^2}\right)$, find the value of the constant a. [2]

(b) Show that

$$\sum_{r=2}^{N} \frac{(n+1)}{n^2(n+2)^2} = p + \frac{q}{(N+1)^2} + \frac{r}{(N+2)^2},$$

where p, q and r are constants to be determined. [3]

(c) Give a reason why the series $\displaystyle\sum_{r=2}^{N} \frac{(n+1)}{n^2(n+2)^2}$ converges as $N \to \infty$, and write down the value of the sum to infinity. [2]

(d) * Deduce the value of

$$\sum_{n=3}^{\infty} \left(\frac{1}{n!} + \frac{(n-1)}{n^2(n-2)^2}\right),$$

giving your answer in exact form. You may use the standard results given in MF15. [4]

9. **[ACJC 17 Prelims]**
By writing

$$\sin\left(x + \frac{1}{4}\right)\pi - \sin\left(x - \frac{3}{4}\right)\pi$$

in terms of a single trigonometric function, find $\displaystyle\sum_{x=1}^{n} \cos\left(x - \frac{1}{4}\right)\pi$, leaving your answer in terms of n. [4]

10. **[IJC 17 Prelims]**
A sequence $u_1, u_2, u_3, \ldots$ is such that

$$u_n = \frac{1}{2n^2(n-1)^2} \text{ and } u_{n+1} = u_n - \frac{2}{n(n-1)^2(n+1)^2}, \text{ for all } n \geq 2.$$

(a) Find $\displaystyle\sum_{n=2}^{N} \frac{2}{n(n-1)^2(n+1)^2}$. [3]

(b) Explain why $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n(n-1)^2(n+1)^2}$ is a convergent series, and state the value of the sum to infinity. [2]

(c) Using your answer in part (a), find $\displaystyle\sum_{n=1}^{N} \frac{2N}{(n+1)^2 n^2 (n+2)^2}$. [2]

11. **[MI 17 Prelims (modified)]**

(a) Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{2}{r(r+2)} = \frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}$$

for all positive integers n. [4]

(b) Explain why $\sum_{r=1}^{n} \frac{2}{r(r+2)}$ is a convergent series, and state the value of the sum to infinity. [2]

(c) Using the result in part (a), find $\sum_{r=5}^{N} \frac{2}{(r-2)(r-4)}$. [2]

12. **[TJC 17 Prelims]**

(a) Given that $\sin\left[(n+1)x\right] - \sin\left[(n-1)x\right] = 2\cos nx \sin x$, show that

$$\sum_{r=1}^{n} \cos rx = \frac{\sin(n+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}.$$

[4]

(b) Hence express

$$\cos^2\left(\frac{x}{2}\right) + \cos^2(x) + \cos^2\left(\frac{3x}{2}\right) + \ldots + \cos^2\left(\frac{11x}{2}\right)$$

in the form $a\left(\frac{\sin bx}{\sin cx} + d\right)$, where a, b, c and d are real numberes. [3]

13. **[TPJC 17 Prelims]**

(a) Express $\frac{1}{r^2-1}$ in partial fractions, and deduce that

$$\frac{1}{r(r^2-1)} = \frac{1}{2}\left(\frac{1}{r(r-1)} - \frac{1}{r(r+1)}\right).$$

[2]

(b) Hence, find the sum, S_n of the first n terms of the series

$$\frac{1}{2\times 3} + \frac{1}{3\times 8} + \frac{1}{4\times 15} + \cdots$$

[4]

(c) Explain why the series converges, and write down the value of the sum to infinity. [2]

(d) Find the smallest value of n for which S_n is smaller than the sum to infinity by less than 0.0025. [3]

Answers

1. $\frac{3}{2}-\frac{2}{n+1}+\frac{1}{n+2}$.
   $\frac{3}{2}-\frac{2}{n+2}+\frac{1}{n+3}$.

2. $1-\frac{1}{(N+1)^3}$.
   $\frac{1}{8}-\frac{1}{N^3}$.
   1.

3. $\frac{5}{99}$.
   $\frac{1}{3}-\frac{n+2}{(2n+3)(2n+5)}$.

4. $\frac{3}{2}-\frac{N+3}{(N+2)!}$.
   $\frac{3}{2}$.
   $n=12$.

5. $\frac{n(n+1)(2n+1)}{6}$.
   $\frac{n^2(n+1)^2}{4}$.

6. $\frac{3}{4}$.

7. $\frac{1}{2}(\frac{5}{6}-\frac{1}{N+2}-\frac{1}{N+3})$.
   $\frac{7}{24}-\frac{1}{2N+4}-\frac{1}{2N+6}$.

8. (a) $a=\frac{1}{4}$.
   (b) $p=\frac{13}{144}, q=-\frac{1}{4}, r=-\frac{1}{4}$.
   (c) $\frac{13}{144}$.
   (d) $\mathrm{e}-\frac{35}{16}$.

9. $\frac{1}{2}\sin(n+\frac{1}{4})\pi-\frac{1}{2\sqrt{2}}$.

10. (a) $\frac{1}{8}-\frac{1}{2N^2(N+1)^2}$.
    (b) $\frac{1}{8}$.
    (c) $\frac{N}{8}\left(1-\frac{4}{(N+1)^2(N+2)^2}\right)$.

11. (b) $\frac{3}{2}$.
    (c) $\frac{3}{2}-\frac{2N-5}{(N-3)(N-2)}$.

12. $\frac{1}{4}\left(\frac{\sin\frac{23}{2}x}{\sin\frac{1}{2}x}+21\right)$.

13. (b) $\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$.
    (c) As $n\to\infty$, $\frac{1}{2(n+1)(n+2)}\to 0$. Hence $\frac{1}{4}-\frac{1}{2(n+1)(n+2)}\to\frac{1}{4}$.
    Sum to infinity $=\frac{1}{4}$.
    (d) 13.