1 P&C

1. **[DHS MYE 18 (modified)]**

   A group of students consisting of two girls and six boys turned up for an interview.

   (a) Eight colour tags are distributed randomly to this group of students, with each student receiving only one tag. There are 1 red, 3 green and 4 yellow tags and each coloured tag is identical.

   i. Find the total number of ways to distribute the tags. [1]

   ii. Find the number of ways to distribute the tags such that the two girls get yellow tags. [1]

   (b) While waiting for their turn for the interview, the students randomly sat on the eight chairs lined up in a row outside the interview room.

   i. Find the number of ways for the two girls to sit adjacent to each other. [2]

   ii. Find the number of ways for the two girls to sit adjacent to each other and on of the boys, Peter, does not sit next to any girl. [2]

   iii. Find the number of ways for Peter to not sit next to any girl. [1]

2. **[EJC MYE 18]**

   A committee consisting of six persons is to be selected from five women and six men. Find the number of possible selections such that

   (a) the chosen committee will contain exactly two men, [1]

   (b) the chosen committee will have more men than women in the committee. [2]

   The chosen committee consists of a married couple together with three other men and one other woman. They are seated round a table for six. Find the number of possible arrangements such that

   (c) The husband is sitting next to his wife, [2]

   (d) the husband is separated from his wife by at least two other people on each of his left and right sides. [2]

3. **[MI MYE 18 (modified)]**

   Each of the digits $1, 2, 3, 4, 5, 6$ is written on a separate card. The cards are then laid out in a row to form a number.

   (a) How many distinct 6-digit numbers can be formed? [1]

   (b) How many distinct 3-digit numbers can be formed? [1]

   An additional card with digit 1 is added to the set of cards to be arranged.

   (c) How many ways are there to form a 7-digit number? [1]

   (d) How many ways are there to form a 7-digit number that starts and ends with the same digit? [1]

4. **[MI MYE 18]**

Three boys, two girls and two teachers sit in a line on stools in a coffee bar.

(a) In how many ways can they arrange themselves so that the two teachers are next to each other and not all the girls are next to each other? [3]

The eight people decide to sit at a round table with ten chairs instead.

(b) Find the number of different possible arrangements. [2]

5. **[RI MYE 18]**

The eleven letters in the word INSTITUTION are individually printed on eleven identical cards.

(a) The eleven cards are arranged in a line.

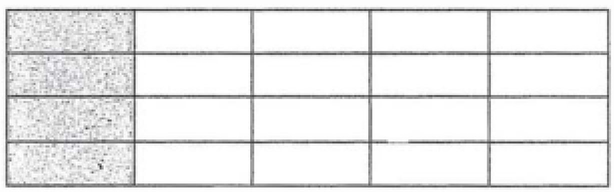
i. Find the number of different arrangements of the eleven cards that can be made. [1]

ii. Find the number of different arrangements that can be made if S,U and O are separated from one another. [2]

(b) Three cards are to be chosen from the eleven cards; the order in which they are chosen does not matter. Find the number of different possible selections of three cards. [3]

6. **[TJC MYE 18]**

As part of a military exercise, a rectangular piece of land is divided into 4 equal rows and 5 equal columns, forming 20 smaller plots as shown in the diagram below. The shaded plots are designated to be "No Man's Land".



3 identical simulated land mines are placed in 3 distinct plots of land. Find the number of ways this can be done if

(a) at least one mine is placed in "No Man's Land", [2]

(b) all the mines are placed in the same column or in the same row, [3]

(c) no two mines are placed in the same column and no two mines are placed in the same row. [2]

2 Functions

7. **[YIJC Prelims 19]**

The function f is defined by $f : x \mapsto 2 + \dfrac{3}{x},\ x \in \mathbb{R}, x > 0$.

(a) Sketch the graph of $y = f(x)$. Hence show that f has an inverse. [2]

(b) Find $f^{-1}(x)$ and state the domain of f^{-1}. [3]

(c) On the same diagram as in part (a), sketch the graphs of $y = f^{-1}(x)$ and $y = f^{-1}f(x)$. [2]

(d) Explain why f^2 exists and find $f^2(x)$. [2]

8. **[TJC Prelims 19]**

The function f is defined by

$$f : x \mapsto \frac{x^2}{2-x},\ x \in \mathbb{R}, 0 \leq x < 2.$$

(a) Find $f^{-1}(x)$ and write down the domain of f^{-1}. [4]

It is given that

$$g : x \mapsto \frac{1}{1+e^{-x}},\ x \in \mathbb{R}, x \geq 0.$$

(b) Show that fg exists. [2]

(c) Find the range of fg. [2]

9. **[NJC Prelims 19]**

Functions f and g are defined by

$$\begin{aligned} &f : x \mapsto 2|x-p|+1 && \text{for } x \in \mathbb{R}, p > 1, \\ &g : x \mapsto x(x-q) && \text{for } x \in \mathbb{R}, q < 0. \end{aligned}$$

(a) Explain why f does not have an inverse. [2]

The domain of f is now restricted to $x \leq k$.

(b) Write down the largest value of k for which the function f^{-1} exists. Hence find $f^{-1}(x)$ and state the domain of f^{-1}. [4]

(c) Sketch on the same diagram the graphs of $y = f(x)$ and $y = f^{-1}(x)$, giving the coordinates of the axial intercepts. Hence solve $f(x) = f^{-1}(x)$. [4]

3 DE

10. **[VJC Prelims 19]**

A ball-bearing is dropped from a point O and falls vertically through the atmosphere. Its speed at O is zero, and t seconds later, its velocity is v ms^{-1} and its displacement from O is x m. The rate of change of v with respect to t is given by $10 - 0.001v^2$.

(a) Show that $v = 100\left(\frac{e^{\frac{t}{5}} - 1}{e^{\frac{t}{5}} + 1}\right)$. [4]

(b) Find the value of v_0, where v_0 is the value approached by v for large values of t. [1]

(c) By using chain rule, form an equation relating $\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}v}{\mathrm{d}t}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$. Given that $v = \frac{\mathrm{d}x}{\mathrm{d}t}$, form a differential equation relating v and x. Show that

$$v = 100\sqrt{1 - e^{-\frac{x}{500}}}.$$

[5]

(d) Find the distance of the ball-bearing from O after 5 seconds, giving your answer correct to 2 decimal places. [3]

11. **[SAJC Prelims 19]**

A water tank contains 1 cubic meter of water initially. The volume of water in the tank at time t seconds is V cubic metres. Water flows out of the tank at a rate proportional to the volume of water in the tank and at the same time, water is added to the tank at a constant rate of k cubic metres per second.

(a) Show that $\frac{\mathrm{d}V}{\mathrm{d}t} = k\left(1 - \frac{a}{k}V\right)$, where a is a positive constant. [2]

(b) Hence find V in terms of t. [5]

(c) Sketch the solution curve for V against t, such that

i. $a < k$,

ii. $a > k$.

For cases (i) and (ii), describe and explain what would happen to the volume of water, V in the tank eventually. [4]

12. **[NJC Prelims 19]**

(a) Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2}{\sqrt{1-4x^2}}, \text{ where } -\frac{1}{2} < x < \frac{1}{2}.$$

[5]

(b) The particular solution $y = f(x)$ has a minimum point at the origin.
Find $f(x)$. [2]

(c) Sketch the graph of this particular solution. [2]

Answers

1. (a) i. 280.
       ii. 60.
   (b) i. 10,080.
       ii. 7200.
       iii. 21,600.
2. (a) 75.
   (b) 181.
   (c) 48.
   (d) 24.
3. (a) 720.
   (b) 120.
   (c) 2,520.
   (d) 120.
4. (a) 8,640.
   (b) 101,440.
5. (a) 554,400.
   (b) 282,240.
   (c) 37.
6. (a) 580.
   (b) 60.
   (c) 240.
7. (a) Since any horizontal line $y = k, k \in \mathbb{R}$ intersects the graph of $y = f(x)$ at most once, f is one-one and it has an inverse.
   (b) $f^{-1}(x) = \frac{3}{x-2}$.
       $D_{f^{-1}} = R_f = (2, \infty)$.
   (d) Since $(2, \infty) - R_f \subseteq D_f = (0, \infty)$.
       $f^2(x) = \frac{7x+6}{2x+3}$.
8. (a) $f^{-1}(x) = \frac{-x+\sqrt{x^2+8x}}{2}$.
       $D_{f^{-1}} = R_f = [0, \infty)$.
   (b) Since $[0, 2) - R_g \subseteq D_f = [\frac{1}{2}, 1)$.
   (c) $R_{fg} = [\frac{1}{6}, 1)$.

9. (a) The line $y = 2p+1$ cuts the graph of $y = f(x)$ twice. Hence f is not a one-one function and its inverse does not exist.

   (b) Largest value of k is p.
   $f^{-1}(x) = p + \frac{1-x}{2}$.
   $D_{f^{-1}} = R_f = [1, \infty)$.

   (c) $x = \frac{2p+1}{3}$.

10. (b) $v_0 = 100$.

    (d) 120.11 m.

11. (b) $V = \frac{1}{a}\left(k - (k-a)e^{-at}\right)$.

    (c) If $a < k$, V increases and approaches $\frac{k}{a}$ cubic meters eventually.
    If $a > k$, V decreases and approaches $\frac{k}{a}$ cubic meters eventually.

12. (a) $y = x\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + Cx + D$.

    (b) $f(x) = x\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} - \frac{1}{2}$.