Module 4: Calculus (Integration, Definite Integrals & DE)

1. [2017/RI/P1/3]

(a) Find $\int \frac{x+2}{\sqrt{(1-8x-4x^2)}}\,\mathrm{d}x.$ [4]

(b) Use the substitution $x = 2\sec\theta$ to find the exact value of $\int_2^4 \frac{1}{x}\sqrt{(x^2-4)}\,\mathrm{d}x$. [4]

2. [2017/SRJC/P1/9]

(a) Using the substitution $u = 2x+3$, find $\int \frac{x}{(2x+3)^3}\,\mathrm{d}x$ in the for $-\frac{Px+Q}{R(2x+3)^2}+c$ where P, Q and R are positive integers to be determined. [3]

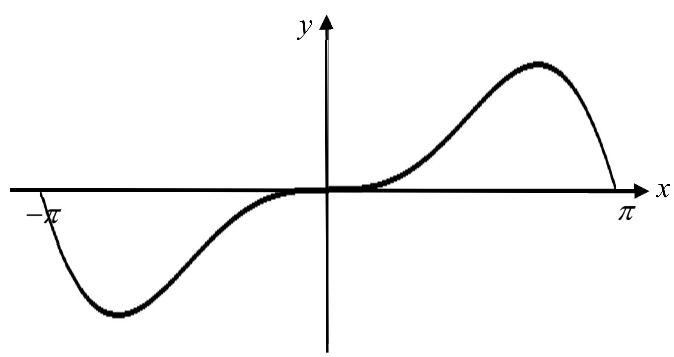
Hence find $\int \frac{x\ln(4x+3)}{(2x+3)^3}\,\mathrm{d}x$. [3]

(b) Find $\int \sin 4x \cos 6x\ \mathrm{d}x$. [2]

Hence or otherwise, find $\int \mathrm{e}^x \sin 4\mathrm{e}^x \cos 6\mathrm{e}^x\ \mathrm{d}x$. [1]

3. [2016/HCI/P1/8]

The curve C (as shown in the diagram below) has equation $y = x^2 \sin x,\ -\pi \le x \le \pi.$



(i) Calculate the exact area of the region R enclosed by C and the x-axis. [4]

(ii) Sketch the curve with equation $(y+1)^2 - 4(x+2)^2 = 1$, showing clearly the coordinates of the turning points and the equation(s) of any asymptote(s). [2]

(iii) Hence find the volume of the solid generated when the region bounded by the 2 curves is rotated through 4 right angles about the x-axis. [4]

4. [2016/HCI/P2/3]

(i) Use the substitution $x = 3\sin\theta + 1,\ 0 < \theta < \frac{\pi}{2}$ to find $\int \frac{x}{\sqrt{9-(x-1)^2}}\,\mathrm{d}x$. [4]

(ii) A curve has parametric equations

$$x = \frac{1}{\sqrt{9-(t-1)^2}}, \quad y = t^2, \quad 0 < t < 4.$$

(a) Sketch the curve, indicating the end point and the equation of the asymptote. [3]

(b) Using the result in part **(i)**, find the exact area of the region bounded by the curve, the lines $y = 1$, $y = \frac{25}{4}$ and $x = \frac{8}{7}$. [4]

5. [2016/NYJC/P1/9(modified)]

Two biologists are investigating the growth of a certain bacteria of size x hundred thousand at time t days. It is known that the number of bacteria initially is 20% of a, where a is a positive constant.

(i) One biologist believes that x and t are related by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = x(a-x)$. Given that the number of bacteria increases to 50% of a when $t = \ln 2$ days, show that $x = \frac{2}{4\mathrm{e}^{-2t}+1}$. [7]

(ii) Another biologist believes that x and t are related by the differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 10 - 9t^2$. Find the general solution of this differential equation. [3]

6. [2012/AJC/P2/4(b)]

A new drug for the treatment of diabetes is administered to a patient at a constant rate of R mg per day. The rate at which the drug is lost from the patient's body is proportional to the square of the amount x (mg) of the drug present in his body at time t (days).

(i) If the amount of drug in the patient remains constant at the instant when it is $2R$ (mg), show that $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{4R^2 - x^2}{4R}$.

Given that, when $x = 0$ when $t = 0$, find x, in terms of R and t. [5]

(ii) Explain the significance of this result in the long run. [1]

Answers:

1. **(a)** $-\frac{1}{4}\sqrt{1-8x-4x^2}+\frac{1}{2}\sin^{-1}\frac{2\sqrt{5}\left(x+1\right)}{5}+C$

(b) $2\left[\sqrt{3}-\frac{\pi}{3}\right]$

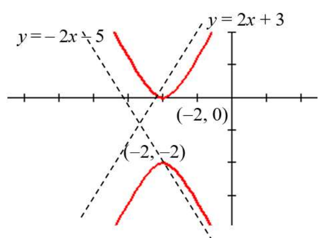
2. **(a)** $P = 4$, $Q = 3$ and $R = 8$

$$\int\frac{x\ln\left(4x+3\right)}{\left(2x+3\right)^3}\,\mathrm{d}x=-\frac{\left(4x+3\right)\ln\left(4x+3\right)+2(2x+3)}{8\left(2x+3\right)^2}+C$$

(b) $-\frac{1}{20}\cos 10x+\frac{1}{4}\cos 2x+C$ **,** $-\frac{1}{20}\cos 10\mathrm{e}^x+\frac{1}{4}\cos 2\mathrm{e}^x+C$

3. **(i)** $2(\pi^2-4)$ units^2

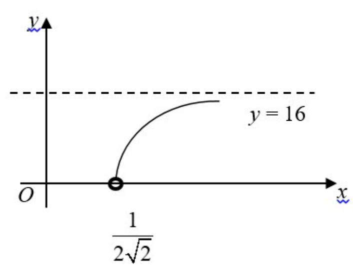
(ii)



(iii) 26.8 units^3

4. **(i)** $-\sqrt{9-(x-1)^2}+\sin^{-1}\frac{x-1}{3}+C$

(i)(a)



(ii)(b) $3\sqrt{3}-\frac{\pi}{3}$

5. **(ii)** $x=5t^2-\frac{3}{4}t^4+At+0.2a$

6. The amount of drug in the body will not exceed $2R$ mg regardless of the period of treatment.