EUNOIA JUNIOR COLLEGE

JC1 Mid-Year Examination 2019

General Certificate of Education Advanced Level

Higher 2

CANDIDATE NAME			
CLASS		INDEX NO.	

MATHEMATICS

9758/01

04 July 2019

Paper 1 [80 marks]

2 hours 30 minutes

Candidates answer on the Question Paper

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and question number on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **28** printed pages (including this cover page) and **2** blank page.

For markers' use:										
Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	**Q7**	**Q8**	**Q9**	**Q10**	**Total**

1 A vegetable store sells three types of vegetables, namely, water cress, long beans and cabbages. The store sells each type of vegetable at a different price per kilogram. Wayne, Andy and Olivia each buys various amounts of each vegetable. However, they cannot remember the individual prices per kilogram, but can remember the total amount that they each paid. The masses of each type of vegetable bought and the total amount paid are shown in the following table.

	Wayne	Andy	Olivia
Water Cress (kg)	2.50	0.75	1.60
Long Beans (kg)	0.50	1.25	0.95
Cabbages (kg)	1.50	0.50	1.05
Total amount paid ($)	33.15	21.46	28.47

Keith visits the same vegetable store and buys 3 kg of water cress, 0.25 kg of long beans and 1.75 kg of cabbages. Assuming that, for each type of vegetable, the price per kilogram paid by Wayne, Andy, Olivia and Keith is the same, determine the amount Keith has to pay. [5]

2 Solve algebrically

$$\frac{x^3+x^2-8x-12}{x^3+4x^2+5x} \le 0 .$$

[5]

3 **(a)** The curve $y = \mathrm{f}(x)$ has asymptotes $x = -1$ and $y = 4$. State the equations of asymptotes of the curve $y = 2\mathrm{f}\left(-x\right) - 3$. [3]

(b) The graph of $y = \mathrm{f}(x)$ undergoes the following sequence of transformations

A: Stretch with scale factor $\frac{2}{3}$ parallel to the x-axis

B: Reflect about the x-axis

C: Translate 4 units in the negative x-direction

Given that the equation of the resulting curve is $y = -\frac{1}{3x+13}$, find the equation of the curve before the 3 transformations were effected. [4]

4 In mathematics, the inequality of arithmetic and geometric means, or more briefly the AM–GM inequality, states that the AM (Arithmetic mean) of a series of real numbers is always greater or equal to the GM (Geometric mean).

The simplest non-trivial case of the AM-GM inequality (i.e. when $n = 2$) states that:

For real numbers a and b such that $a \geq 0$ and $b \geq 0$,

$$\frac{a+b}{2} \geq \sqrt{ab}$$

and that equality holds if and only if $a = b$.

(i) By considering $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$, prove that $\frac{a+b}{2} \geq \sqrt{ab}$. [2]

(ii)

Functions f and g are defined by

$$f(x) = 9x\sin x + \frac{4}{x\sin x} \qquad 0 < x < \pi,$$

$$g(x) = \ln x \qquad x > 5.$$

Use the AM-GM inequality to show that $f(x) \geq 12$. [2]

(iii) Determine whether the composite function gf exists. [2]

5 Given that $\frac{2}{r(r^2-1)} = \frac{1}{(r-1)} - \frac{2}{r} + \frac{1}{(r+1)}$.

(i) Find $\sum_{r=2}^{n} \frac{2}{r(r^2-1)}$.(There is no need to express your answer as a single algebraic fraction.)

[3]

(ii) Give a reason why the series $\sum_{r=2}^{\infty} \frac{2}{r(r^2-1)}$ converges, and write down its value. [2]

(iii) Use your answer in part **(i)** to find $\sum_{r=10}^{n} \frac{2}{r(r+1)(r+2)}$. [3]

6 The curve C has parametric equations

$$x=-\frac{1}{2}\cos 2\theta, \quad y=\sqrt{2}\sin\theta-1, \text{ where } -\pi<\theta<\pi.$$

(i) Sketch the curve C, giving the coordinates of its vertex, endpoints and any points where C crosses the x- and y- axes. [5]

(ii) The line $y = 2x - 1$ intersects C. Find the point(s) of intersection. [3]

7 The curve C has equation $y = \frac{(x-2)^2}{x+4}$.

(i) Find, using an algebraic method, the set of values that y can take. [3]

(ii) Sketch the curve C, stating the coordinates of any points of intersection with the axes, any turning points, and the equations of any asymptotes. [4]

(iii) By drawing $x^2 + y^2 = 1$ on the same diagram, deduce the number of real roots for the equation

$$x^2 + \left[\frac{(|x|-2)^2}{|x|+4}\right]^2 = 1.$$

[2]

8 Functions f and g are defined by

$$\mathrm{f}(x)=\begin{cases} x(2-x) & \text{for } 0 \le x < 1, \\ 2-x & \text{for } 1 \le x < 2, \end{cases}$$

$$\mathrm{g}(x)=\mathrm{e}^x, \quad x \in \mathbb{R},\ x \ge 0.$$

(i) Sketch the graph of $y=\mathrm{f}(x)$ for $0 \le x < 2$. [3]

(ii) Give a definition (including the domain) of the composite function gf and its range. [3]

(iii) Given further that $\mathrm{f}(x-2)=\mathrm{f}(x)$ for all real values of x, find the exact value of $\mathrm{f}\left(\frac{101}{4}\right)$. [2]

9 The function f is defined as

$$\mathrm{f}: x \mapsto \sqrt{x+1} - \frac{1}{2}, \qquad x \in \mathbb{R},\ x > -1.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$. Your sketch should state the coordinates of any points of intersection with the axes and endpoints. [2]

(ii) Find $f^{-1}(x)$, stating the domain of f^{-1}. [3]

(iii) On the same diagram as in part **(i)**, sketch the graph of $y = f^{-1}(x)$. [1]

(iv) Write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$, and hence find the exact solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [4]

(v) Using (iii) and (iv) to deduce the solution set of $\mathrm{f}(x) \geq \mathrm{f}^{-1}(x)$. [2]

10 A painter is tasked to paint the interior of a HDB flat. First, the painter fills up paint cans each having a volume of 2 litres. To prevent spillage, the tap used to fill the paint cans has a special mechanism that can control the volume of paint poured into the paint can every second. In the 1^{st} second, 100 ml of paint is poured into the paint can and for each subsequent second, the volume of paint poured into the paint can is 5% less than the volume in the previous second.

(i) Find the volume of paint poured into the paint can in the 24^{th} second. [1]

(ii) Find, to the nearest ml, the total volume of paint in the paint can after 1 minute. [2]

(iii) Explain why the paint will never overflow the paint can. [2]

At the start of each day of painting, the painter paints an area of $70\ \text{m}^2$ in the 1^{st} hour. In the subsequent hours, the fatigue of painting results in the painter painting $3\ \text{m}^2$ less compared to the previous hour. For each day, the painter continuously paints for only 10 hours.

(iv) In which hour of each day does the painter paint an area of exactly $49\ \text{m}^2$? [2]

(v) Find the total area painted in a day. [1]

(vi) The painter takes up a project to paint a house with an interior area of 2045 m^2. Find the number of complete hours the painter takes to finish painting. [4]

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2019 J1 MYE Suggested solution with Marker's Comments

1 A vegetable store sells three types of vegetables, namely, water cress, long beans and cabbages. The store sells each type of vegetable at a different price per kilogram. Wayne, Andy and Olivia each buys various amounts of each vegetable. However, they cannot remember the individual prices per kilogram, but can remember the total amount that they each paid. The masses of each type of vegetable bought and the total amount paid are shown in the following table.

	Wayne	Andy	Olivia
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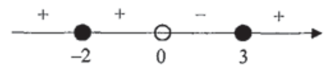
Keith visits the same vegetable store and buys 3 kg of water cress, 0.25 kg of long beans and 1.75 kg of cabbages. Assuming that, for each type of vegetable, the price per kilogram paid by Wayne, Andy, Olivia and Keith is the same, determine the amount Keith has to pay. [5]

Suggested solution	Comments
Let x, y, z be the cost per kilogram of Water Cress, Long Beans and Cabbages respectively.	A significant number of candidates did not define their variables properly and correctly.
$2.5x+0.5y+1.5z=33.15$ $0.75x+1.25y+0.5z=21.46$ $1.6x+0.95y+1.05z=28.47$	
From GC, we have $x=4.82$, $y=9.98$, $z=10.74$	
Total cost of Keith's vegetables = $3(4.82)+0.25(9.98)+1.75(10.74)=35.75$ Hence Keith has to pay $ 35.75.	A few candidates wrongly rounded their final answer for this context.

2 Solve algebrically

$$\frac{x^3+x^2-8x-12}{x^3+4x^2+5x}\le 0.$$

[5]

Suggested solution	Comments
$\frac{x^3+x^2-8x-12}{x^3+4x^2+5x}\le 0$ $\frac{(x-3)(x+2)^2}{x(x^2+4x+5)}\le 0$ $\frac{(x-3)(x+2)^2}{x\left[(x+2)^2+1\right]}\le 0,\quad (x+2)^2+1>0 \quad \forall x\in\mathbb{R}$ OR since the coefficient of x^2 in x^2+4x+5 is positive and discriminant is $4^2-4(1)(5)=-4<0$, $x^2+4x+5>0 \quad \forall x\in\mathbb{R}$. $\frac{(x-3)(x+2)^2}{x}\le 0$  $x=-2$ **or** $0<x\le 3$	Most candidates are able to factorise completely the expression. *Candidates need to explain $x^2+4x+5>0$ or $(x+2)^2+1>0$ as part of the algebraic working. Common mistake: Many candidates missed out the solution $x=-2$.

3 **(a)** The curve $y = \mathrm{f}(x)$ has asymptotes $x = -1$ and $y = 4$. State the equations of asymptotes of the curve $y = 2\mathrm{f}(-x) - 3$. [3]

(b) The graph of $y = \mathrm{f}(x)$ undergoes the following sequence of transformations

A: Stretch with scale factor $\frac{2}{3}$ parallel to the x-axis

B: Reflect about the x-axis

C: Translate 4 units in the negative x-direction

Given that the equation of the resulting curve is $y = -\frac{1}{3x+13}$, find the equation of the curve before the 3 transformations were effected. [4]

Suggested solution	Comments
For $x=-1$, consider $x \leftrightarrow -x \Rightarrow x = 1$ For $y=4$, Method I (scaling followed by translation) consider $y \leftrightarrow \frac{y}{2} \Rightarrow \frac{y}{2} = 4 \Rightarrow y = 8$ $y \leftrightarrow y+3 \Rightarrow y+3 = 8 \Rightarrow y = 5$ Method II(translation followed by scaling) consider $y \leftrightarrow y+\frac{3}{2} \Rightarrow y+\frac{3}{2} = 4 \Rightarrow y = \frac{5}{2}$ $y \leftrightarrow \frac{y}{2} \Rightarrow \frac{y}{2} = \frac{5}{2} \Rightarrow y = 5$ $y \leftrightarrow y+3 \Rightarrow y+3 = 8 \Rightarrow y = 5$ Method III (do 2 steps directly): note that $y = 2\mathrm{f}(-x) - 3 \Rightarrow \frac{y+3}{2} = \mathrm{f}(-x)$ So we'll consider $\frac{y+3}{2} = 4 \Rightarrow y = 5$	Well-attempted, with most students scoring at least 4 marks out of 7 for the entire question. Essential workings must be shown in how candidates arrived at the final answer, as it was possible to obtain the final answer of $y=5$ with the wrong approach $\left(y = \frac{4}{2} + 3 = 5\right)$. Candidates are strongly encouraged to adopt Method I when handling a combination of scaling and translation transformations, i.e. scaling first, followed by translation. Many of the candidates who adopted Method II translated by the wrong number of units. Candidates should not assume an expression for f(x) unless they are sure that such an approach is without any loss in generality.

	Some candidates sketched the two asymptotes and showed the changes in the position of these two asymptotes as they went through the transformations. The sketches are part of the working, and should be drawn properly and neatly in such an approach.
(b)Reversing the transformations on $y = \mathrm{g}(x)$, we have $y = \mathrm{g}(x) \rightarrow y = \mathrm{g}(x-4)$ *C*': [replace x by $(x-4)$] [translation of 4 units in the positive x-direction] $\rightarrow y = -\frac{1}{3(x-4)+13} = -\frac{1}{3x+1}$ *B*': [replace y by $-y$] [reflection about the x-axis] $\rightarrow -y = -\frac{1}{3x+1}$ $\rightarrow y = \frac{1}{3x+1}$ *A*': [replace x by $\frac{2}{3}x$] [scaling with scale factor $\frac{3}{2}$ parallel to the x-axis] $\rightarrow y = \frac{1}{3\left(\frac{2}{3}x\right)+1}$ $\rightarrow y = \frac{1}{2x+1}$ The original equation is $y = \frac{1}{2x+1}$	Common errors made by candidates:- • Replacing x by $-x$ when reflecting about the x-axis • Replacing x by $\frac{3}{2}x$ when scaling by factor $\frac{3}{2}$ parallel to the x-axis Candidates should be using the correct terms in describing the transformations. The following terms in italics are not acceptable:- • *Moving* instead of **translating** • *Squeezing /compressing* instead of **scaling** • *Scaling by factor -1* instead of **reflecting**

4 In mathematics, the inequality of arithmetic and geometric means, or more briefly the AM–GM inequality, states that the AM (Arithmetic mean) of a series of real numbers is always greater or equal to the GM (Geometric mean).

The simplest non-trivial case of the AM-GM inequality (i.e. when $n = 2$) states that:

For real numbers a and b such that $a \geq 0$ and $b \geq 0$,

$$\frac{a+b}{2} \geq \sqrt{ab}$$

and that equality holds if and only if $a = b$.

(i) By considering $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$, prove that $\frac{a+b}{2} \geq \sqrt{ab}$. [2]

(ii)

Functions f and g are defined by

$$\mathrm{f}(x) = 9x\sin x + \frac{4}{x\sin x} \qquad 0 < x < \pi,$$

$$\mathrm{g}(x) = \ln x \qquad x > 5.$$

Use the AM-GM inequality to show that $\mathrm{f}(x) \geq 12$. [2]

(iii) Determine whether the composite function gf exists. [2]

Suggested solution	Comments
(i) $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0 \Rightarrow a - 2\sqrt{ab} + b \geq 0$ $\Rightarrow a + b \geq 2\sqrt{ab}$ $\Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$	This part is well done, except for a handful of students who expanded $\left(\sqrt{a}-\sqrt{b}\right)^2$ wrongly.
(ii) $\mathrm{f}(x) = 9x\sin x + \frac{4}{x\sin x}$ Let $a = 9x\sin x$ and $b = \frac{4}{x\sin x}$. Since $0 < x < \pi$, $a, b \geq 0$. Using the AM-GM inequality, $\frac{a+b}{2} \geq \sqrt{ab}$	This part is badly done. Most students were not able to choose the correct expressions for a and b properly. Some students tried to prove without using the AM-GM inequality which is not allowed in this question.

$\frac{9x\sin x + \frac{4}{x\sin x}}{2} \geq \sqrt{(9x\sin x)\left(\frac{4}{x\sin x}\right)}$ $\frac{\mathrm{f}(x)}{2} \geq \sqrt{36} = 6$ $\mathrm{f}(x) \geq 12$ [shown]	Some students wrongly thought that $\sqrt{36} = \pm 6$ and obtained $\mathrm{f}(x) \geq \pm 12$. Even if they subsequently rejected -12, the students were penalised for this major conceptual error.
(iii) From (ii), $R_f \subseteq [12, \infty) \subseteq D_g = (5, \infty)$ Hence **gf does exists.**	Many students were able to guess that $R_f = [12, \infty)$ and did this part correctly. However, from (ii), one can only conclude that $R_f \subseteq [12, \infty)$, which is sufficient for our purposes. It is not obvious that we can have $\mathrm{f}(x) = 12$ or for $\mathrm{f}(x)$ to tend to infinity. There were also many students who did not know how to check that a composite function exists, and should revise this concept again.

5 Given that $\dfrac{2}{r(r^2-1)}=\dfrac{1}{(r-1)}-\dfrac{2}{r}+\dfrac{1}{(r+1)}$.

(i) Find $\displaystyle\sum_{r=2}^{n}\frac{2}{r(r^2-1)}$. (There is no need to express your answer as a single algebraic fraction.) [3]

(ii) Give a reason why the series $\displaystyle\sum_{r=2}^{\infty}\frac{2}{r(r^2-1)}$ converges, and write down its value. [2]

(iii) Use your answer in part (i) to find $\displaystyle\sum_{r=10}^{n}\frac{2}{r(r+1)(r+2)}$. [3]

Suggested solution	Comments
(i) $\displaystyle\sum_{r=2}^{n}\frac{2}{r(r^2-1)}=\sum_{r=2}^{n}\left[\frac{1}{r-1}-\frac{2}{r}+\frac{1}{r+1}\right]$ $=\left[\begin{array}{l}\frac{1}{1}-\frac{2}{2}+\frac{1}{3}\\+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\\+\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\\+\frac{1}{4}-\frac{2}{5}+\frac{1}{6}\\+\ldots\\+\frac{1}{n-3}-\frac{2}{n-2}+\frac{1}{n-1}\\+\frac{1}{n-2}-\frac{2}{n-1}+\frac{1}{n}\\+\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}\end{array}\right]$ $=\frac{1}{1}-\frac{2}{2}+\frac{1}{2}+\frac{1}{n}-\frac{2}{n}+\frac{1}{n+1}$ $=\frac{1}{2}-\frac{1}{n}+\frac{1}{n+1}$	Generally well done. Most students are able to see the need to use the given expression at the start of the question. **Common Errors:** 1. Changing the lower bound to r = 1, which invalidate the whole working as the given expression $\frac{2}{r(r^2-1)}$ is undefined when r = 1. 2. Listing less than 3 rows in the last part of M.O.D. does not show how you could get the remaining terms. 3. Quite a handful of students wrote in the last 3 rows, that the 2 is missing from the numerator, thus making errors in their final answer.
(ii) $\displaystyle\sum_{r=2}^{n}\frac{2}{r(r^2-1)}=\frac{1}{2}-\frac{1}{n}+\frac{1}{n+1}$ As $n\to\infty,\frac{1}{n}\to 0$ and $\frac{1}{n+1}\to 0$, therefore $\displaystyle\sum_{r=2}^{\infty}\frac{2}{r(r^2-1)}=\frac{1}{2}$ Since $\displaystyle\sum_{r=2}^{\infty}\frac{2}{r(r^2-1)}$ converges to a constant (or finite) number, hence it is convergent.	A lot of students are not able to provide a proper explanation on why the series converges or how they get the value. **Common errors:** 1. Some students explained that $\frac{2}{r(r^2-1)}\to 0$ thus $\displaystyle\sum_{r=2}^{\infty}\frac{2}{r(r^2-1)}=\frac{1}{2}$

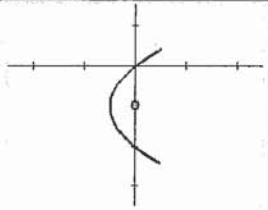
	2. $n\to\infty,\frac{1}{n}\to\infty$ and $\frac{1}{n+1}\to\infty$, thus value is $\frac{1}{2}$.
(iii) $\displaystyle\sum_{r=10}^{n}\frac{2}{r(r+1)(r+2)}=\sum_{r-1=10}^{r-1=n}\frac{2}{(r-1)(r)(r+1)}$ (replace r by $r-1$) $\displaystyle=\sum_{r=11}^{n+1}\frac{2}{(r-1)(r)(r+1)}$ $\displaystyle=\sum_{r=2}^{n+1}\frac{2}{(r-1)(r)(r+1)}-\sum_{r=2}^{10}\frac{2}{(r-1)(r)(r+1)}$ $\displaystyle=\sum_{r=2}^{n+1}\frac{2}{(r-1)(r)(r+1)}-\sum_{r=2}^{10}\frac{2}{(r-1)(r)(r+1)}$ $=\left(\frac{1}{2}-\frac{1}{n+1}+\frac{1}{n+2}\right)-\left(\frac{1}{2}-\frac{1}{10}+\frac{1}{11}\right)$ $=\frac{1}{110}-\frac{1}{n+1}+\frac{1}{n+2}$	Most students are able to see the need to do a replacement of the variable, by making the connection to the expression in (i). **Common Errors:** 1. When the replacement of the variable is done, the following are observed: - The upper and lower bounds are either changed wrongly or not changed at all. - The expression is not changed at all, i.e. $\displaystyle\sum_{r=10}^{n}\frac{2}{r(r+1)(r+2)}=\sum_{r-1=10}^{r-1=n}\frac{2}{r(r+1)(r+2)}$ 2. In finding the answer for $\displaystyle\sum_{r=2}^{10}\frac{2}{(r-1)(r)(r+1)}$, some students added manually one term at a time and ended up with a wrong value due to carelessness. 3. When students are to evaluate $\displaystyle\sum_{r=11}^{n+1}\frac{2}{(r-1)(r)(r+1)}$, some of the students expressed the upper bound of the second summation wrongly, i.e. $\displaystyle=\sum_{r=11}^{n+1}\frac{2}{(r-1)(r)(r+1)}$ $\displaystyle=\sum_{r=2}^{n+1}\frac{2}{(r-1)(r)(r+1)}-\sum_{r=2}^{9}\frac{2}{(r-1)(r)(r+1)}$

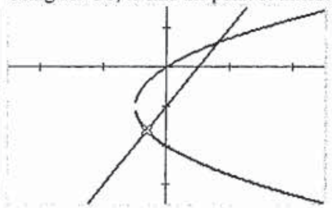
6 The curve C has parametric equations

$$x=-\frac{1}{2}\cos 2\theta,\quad y=\sqrt{2}\sin\theta-1,\text{ where } -\pi<\theta<\pi .$$

(i) Sketch the curve C, giving the coordinates of its vertex, endpoints and any points where C crosses the x- and y- axes. [5]

(ii) The line $y=2x-1$ intersects C. Find the point(s) of intersection. [3]

	Suggested Solutions		**Comments**
(i)	 Find x intercepts $\sqrt{2}\sin\theta-1=0$ $\sin\theta=\frac{1}{\sqrt{2}}$ $\theta=\frac{\pi}{4},\frac{3\pi}{4}$ $x=0$ $\therefore (0,0)$ Find y intercepts $-\frac{\cos 2\theta}{2}=0$ $\cos 2\theta=0$ $2\theta=-\frac{3\pi}{2},-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2}$ $\theta=-\frac{3\pi}{4},-\frac{\pi}{4},\frac{\pi}{4},\frac{3\pi}{4}$ $y=-2,0$ $\therefore (0,-2)$ and $(0,0)$	To find vertex $-1\le\cos 2\theta\le 1$ $\min x=-\frac{1}{2}$ $-\frac{\cos 2\theta}{2}=-\frac{1}{2}$ $\cos 2\theta=1$ $2\theta=0$ $\theta=0$ $y=\sqrt{2}\sin(0)-1=-1$ $\therefore\left(-\frac{1}{2},-1\right)$ To find right end point $\max x=\frac{1}{2}$ $-\frac{\cos 2\theta}{2}=\frac{1}{2}$ $\cos 2\theta=-1$ $2\theta=-\pi,\pi$ $\theta=-\frac{\pi}{2},\frac{\pi}{2}$ $y=\sqrt{2}\sin\left(-\frac{\pi}{2}\right)-1=-\sqrt{2}-1$ $y=\sqrt{2}\sin\left(\frac{\pi}{2}\right)-1=\sqrt{2}-1$ $\therefore\left(\frac{1}{2},\sqrt{2}-1\right)$ and $\left(\frac{1}{2},-\sqrt{2}-1\right)$ Also accept $(0.5,0.414)$ and $(0.5,-2.41)$	Well attempted. Many students wrongly placed ° at both end-points of the graph. A handful of students input $y=\sqrt{2}\sin(\theta-1)$ into the GC, resulting in the wrong graph drawn.

(ii)	$y=2x-1$ By Substitution, $\sqrt{2}\sin\theta-1=2\left(-\frac{\cos 2\theta}{2}\right)-1$ $\sqrt{2}\sin\theta-1=-\cos 2\theta-1$ $\sqrt{2}\sin\theta=-\cos 2\theta$ $\sqrt{2}\sin\theta=-1+2\sin^2\theta$ $2\sin^2\theta-\sqrt{2}\sin\theta-1=0$ $\sin\theta=-0.4370160244$ $\theta=-2.6893142,-0.45227844$ $x=-0.309$ $y=-1.62$	Poorly attempt. Many students were able to perform proper substitution to get the equation $\sqrt{2}\sin\theta=-1+2\sin^2\theta$. However, many subsequently solve for $\sin\theta$, instead of θ, and substituted the values of $\sin\theta$ into the parametric equations to find the coordinates.
	Alternative Method $\cos 2\theta=1-2\sin^2\theta$ By Substitution, $-2x=1-(y+1)^2$ $(y+1)^2=1+2x$ $y+1=\pm\sqrt{1+2x}$ $y=-1\pm\sqrt{1+2x}$ Using the GC, to find the point of intersection,  $(-0.309,-1.62)$ $(0.809,0.618)$ is not intersection point as it is not in the range of $-\pi<\theta<\pi$	For students who attempted this question using the alternative method, many did not reject the intersection point $(0.809,0.618)$.

7 The curve C has equation $y=\frac{(x-2)^2}{x+4}$.

(i) Find, using an algebraic method, the set of values that y can take. [3]

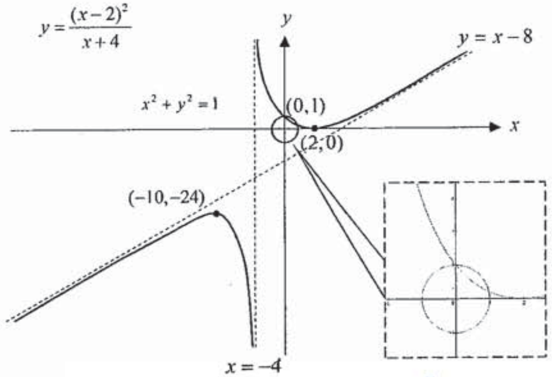
(ii) Sketch the curve C, stating the coordinates of any points of intersection with the axes, any turning points, and the equations of any asymptotes. [4]

(iii) By drawing $x^2+y^2=1$ on the same diagram, deduce the number of real roots for the equation

$$x^2+\left[\frac{(|x|-2)^2}{|x|+4}\right]^2=1.$$

[2]

Suggested solution	Comments
(i) $y=\frac{(x-2)^2}{x+4}=\frac{x^2-4x+4}{x+4}$ $yx+4y=x^2-4x+4$ $x^2-4x+4-yx-4y=0$ $x^2-(4+y)x+4-4y=0$ [Remember the inequality for the discriminant for "can take"] Discriminant ≥ 0 $[-(4+y)]^2-4(1)(4-4y)\geq 0$ $y^2+8y+16-16+16y\geq 0$ $y^2+24y\geq 0$ $y(y+24)\geq 0$ $y\leq -24$ or $y\geq 0$ [Do not use "comma" / "and"! Only "or" is allowed!] $\{y\in\mathbb{R}: y\leq -24 \text{ or } y\geq 0\}$ Also accepted: $y\in(-\infty,-24]\cup[0,\infty)$ [An alarming number of students expanded this wrongly: $[-(4+y)]^2\neq -(4+y)^2$ $[-(4+y)]^2=[-4-y]^2\neq y^2-2(-4)(-y)+4^2$]	This method is standard and fixed for questions of this form "find set of values that ___ can/cannot take" Common but silly mistakes: • Expansion of $(x-2)^2$ • Positive $4y$ on LHS brought to RHS and still positive • Solving $y+24\leq 0$ to get $y\leq 24$ • Expansion of $[-(4+y)]^2$ to get $y^2-8y+16$ • Expansion of $-4(1)(4-4y)$ to get $-16-16y$ Students with "conceptual error", please see your tutors to get this sorted out. Most common error is forgetting to write in set notation. Students should be aware that since the next part of the question is to sketch the graph, it is a big hint that the solution here will be verified by the y-values that the graph in part (ii) can take

(ii)  Students who only drew the top part should realise by now that when the graph is $y=\frac{\text{Quadratic}}{\text{Linear}}$, there will a vertical and an oblique asymptote, splitting the space into 4 regions. The graph will occupy 2 opposite regions.	This question was marked strictly especially penalising students who did not following instructions to write in coordinates form. Some errors to highlight: • Poor long division • $y=x-8$ was drawn with an extremely gentle slope affecting the overall shape • $(0,1)$ written as $(1,0)$ • $y=x-8$ written as $x=4$ Most common errors are: • Turning point (2, 0) is not easily found using GC due to the limitations of the GC, however it is very clear from the numerator that when $x=2, y=0$ • Graph does not tend to asymptotes

Students should take note that since the question states "on the same diagram", it would make sense to sketch the circle on part (ii) instead of re-sketching it in part (iii), students will not be given credit if they sketched the modulus graph with the circle as this is not the intent of the question. Students who identified the correct shape (circle) and radius found it challenging to sketch it in its accurate relative position in part (ii).	Students are to note that when questions require superimposition (meaning two graphs in one diagram), it is important that the relative position to each other is considered. Especially since part (iii) requires a circle, the scale of the axes should be as equal as possible so that the shape drawn actually looks like a circle and not an ellipse.
(iii) Sketch the circle with centre (0,0) and radius of 1 unit. It is symmetrical in the y-axis. Since it cuts the graph of $y=\frac{(x-2)^2}{x+4}$ at one point for $x>0$ and at (0, 1) at y-axis, it will cut the graph of $y=\frac{(\lvert x\rvert-2)^2}{\lvert x\rvert+4}$ at three points, by symmetry. Thus, there are three real roots for the equation.	Since this is a deduce question (same as hence), students are expected to use part (ii)'s graph to interpret the number of real roots to the equation and should not be solving it using a GC.

8 Functions f and g are defined by

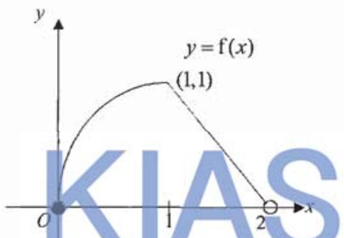
$$f(x)=\begin{cases} x(2-x) & \text{for } 0\le x<1,\\ 2-x & \text{for } 1\le x<2,\end{cases}$$

$$g(x)=e^{x},\quad x\in\mathbb{R},\ x\ge 0.$$

(i) Sketch the graph of $y=f(x)$ for $0\le x<2$. [3]

(ii) Give a definition (including the domain) of the composite function gf and its range. [3]

(iii) Given further that $f(x-2)=f(x)$ for all real values of x, find the exact value of $f\left(\frac{101}{4}\right)$. [2]

Suggested solution	**Comments**
(i)	
	Most students are able to give the correct diagram, but needs improvement. • Draw bigger diagram. • Ensure all details are given, such as labelling important points $(0,0)$, $(1,1)$, $(2,0)$ and putting an open circle at $(2,0)$. • Use ruler to draw all straight lines.
(ii) $R_f=[0,1]$ $D_g=[0,\infty)$ $gf(x)=\begin{cases} e^{x(2-x)} & \text{for } 0\le x<1,\\ e^{2-x} & \text{for } 1\le x<2.\end{cases}$ $R_{gf}=[1,e]$	Most students are able to give the function gf correctly, except some poor presentation observed. The function gf is a piecewise function and domain should be written in inequality form behind the rule. The range of gf is **not** to be in piecewise but as a whole. i.e. $R_{gf}=[1,e]$
(iii) $f\left(25\frac{1}{4}\right)$ $=f\left(25\frac{1}{4}-2\right)$	This part is not well done because some did not know how to approach this question and often left blank.

$=f\left(23\frac{1}{4}\right)$ $=f\left(21\frac{1}{4}\right)=\cdots=f\left(\frac{5}{4}\right)=2-\frac{5}{4}=\frac{3}{4}$	Students ought to read up on periodic functions from tutorials and attempt to reduce $25\frac{1}{4}$ to the given domain. And since $x=\frac{5}{4}$ falls into the domain of $1\le x<2$, use $f(x)=2-x$ $\therefore f\left(\frac{5}{4}\right)=2-\frac{5}{4}=\frac{3}{4}$

9 The function f is defined as

$$f : x \mapsto \sqrt{x+1} - \frac{1}{2}, \qquad x \in \mathbb{R},\ x > -1.$$

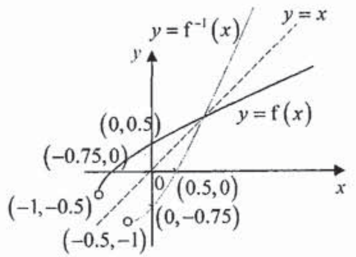
(i) Sketch the graph of $y = f(x)$. Your sketch should state the coordinates of any points of intersection with the axes and endpoints. [2]

(ii) Find $f^{-1}(x)$, stating the domain of f^{-1}. [3]

(iii) On the same diagram as in part **(i)**, sketch the graph of $y = f^{-1}(x)$. [1]

(iv) Write down the equation of the line in which the graph of $y = f(x)$ must be reflected in order to obtain the graph of $y = f^{-1}(x)$, and hence find the exact solution of the equation $f(x) = f^{-1}(x)$. [4]

(v) Using (iii) and (iv) to deduce the solution set of $f(x) \geq f^{-1}(x)$. [2]

Suggested solution	**Comments**
(i) and (iii)  	(i)Many students are able to sketch the graph of $y = f(x)$ but missed out the x-intercepts $(-0.75, 0)$. The question asked for "coordinates of any points of intersection with the axes and endpoints" but most of them are not able to provide their answers in that form. iii) Wasn't well done. Common mistakes are : • Their $y = f^{-1}(x)$ curve bend towards to y-axis. Such shape doesn't even warrant a function let alone a 1-1 function. • $y = f^{-1}(x)$ are supposed to be the reflection of the curve about y = x. Many of them didn't show that in their combined graph. • A couple of student put $(-\infty, \infty)$ as one of the end-point which is incorrect.

(ii) $y = f(x)$ $y = \sqrt{x+1} - \frac{1}{2}$ $y + \frac{1}{2} = \sqrt{x+1}$ $x = \left(y + \frac{1}{2}\right)^2 - 1$ $\therefore f^{-1}(x) = \left(x + \frac{1}{2}\right)^2 - 1$ $D_{f^{-1}} = (-\frac{1}{2}, \infty), R_{f^{-1}} = (-1, \infty)$	(ii) This part is well done. Most student are able to get the rule of inverse. Some student are not able to state the Domain of the f inverse despite sketching $y = f(x)$ correctly in (i).
(iv) **The line is** $y = x$ $f(x) = f^{-1}(x)$ From the sketch, can consider solving **Method 1.** $f^{-1}(x) = x \Rightarrow \left(x + \frac{1}{2}\right)^2 - 1 = x \Rightarrow x^2 + x + \frac{1}{4} - 1 = x$ $\Rightarrow x^2 = \frac{3}{4}$ $\Rightarrow x = \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$ [rejected as $x > -0.5$] **Method 2.** From the sketch, can consider solving $f(x) = x \Rightarrow \sqrt{x+1} - \frac{1}{2} = x \Rightarrow \sqrt{x+1} = x + \frac{1}{2}$ $\Rightarrow x + 1 = x^2 + x + \frac{1}{4} \Rightarrow x^2 = \frac{3}{4}$ $\Rightarrow x = \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$ [rejected as $x > -0.5$]	(iv)Almost all student are able to state the equation $y = x$. Some student used both the equations of $f(x)$ and $f^{-1}(x)$ **directly** to solve $f(x) = f^{-1}(x)$. Unfortunately the question (**HENCE**) requires them to use $y = x$ to get the intersection. Among the reasons of rejecting $x = -\frac{\sqrt{3}}{2}$, only a couple of them are able to give the correct reason $x > -0.5$ which is the intersection of the 2 domains of $f(x)$ and $f^{-1}(x)$.
(v) From the sketch. $\left\{x \in \mathbb{R}, -0.5 < x \leq \frac{\sqrt{3}}{2}\right\}$	(v)Many student gave $-1 < x \leq \frac{\sqrt{3}}{2}$ which was incorrect as $f^{-1}(x)$ will not be defined in $-1 < x \leq -0.5$

10 A painter is tasked to paint the interior of a HDB flat. First, the painter fills up paint cans each having a volume of 2 litres. To prevent spillage, the tap used to fill the paint cans has a special mechanism that can control the volume of paint poured into the paint can every second. In the 1st second, 100 ml of paint is poured into the paint can and for each subsequent second, the volume of paint poured into the paint can is 5% less than the volume in the previous second.

(i) Find the volume of paint poured into the paint can in the 24th second. [1]

(ii) Find, to the nearest ml, the total volume of paint in the paint can after 1 minute. [2]

(iii) Explain why the paint will never overflow the paint can. [2]

At the start of each day of painting, the painter paints an area of 70 m^2 in the 1st hour. In the subsequent hours, the fatigue of painting results in the painter painting 3 m^2 less compared to the previous hour. For each day, the painter continuously paints for only 10 hours.

(iv) In which hour of each day does the painter paint an area of exactly 49 m^2? [2]

(v) Find the total area painted in a day. [1]

(vi) The painter takes up a project to paint a house with an interior area of 2045 m^2. Find the number of complete hours the painter takes to finish painting. [4]

Suggested solution	Comments
(i) Volume of paint poured at 24th second $=100(0.95)^{24-1}=30.736\approx 30.7$ ml (3s.f.)	Well attempted. Able to use the formula $u_n=ar^{n-1}$ where $r\neq 0$
(ii) Total volume of paint in the paint can after 1 min (60 sec) $=\dfrac{100\left[1-(0.95)^{60}\right]}{1-0.95}=1907.86\approx 1908$ ml (nearest ml)	Well attempted. Able to use the formula • $S_n=\dfrac{a(r^n-1)}{r-1},\ r>1$ $=\dfrac{a(1-r^n)}{1-r},\ r<1$ • Careless in question reading, did not leave answer in nearest ml

(iii) Considering the sum to infinity for GP, $\dfrac{100}{1-0.95}=2000$ Therefore, the greatest theoretical volume of paint that the paint can will be filled is 2 litres, hence the paint will not overflow.	Able to apply: • the geometric series is convergent if $\lvert r\rvert<1$ and the sum to infinity exists and is given by $S_\infty=\dfrac{a}{1-r}$
(iv) $70+(n-1)(-3)=49$ $n=7+1$ $=8$ The painter will paint an area of 49 m^2 at the 8th hour.	• Majority are able to apply the formula: $u_n=a+(n-1)d$ • Some use the wrong formula: $S_n=\dfrac{n}{2}\left[2a+(n-1)d\right]$ • Careless in calculation assuming d=3 • Some answer by listing which is not recommended
(v) Total area painted after 10 hours $=\dfrac{10}{2}\left[2(70)+(10-1)(-3)\right]$ $=565$ m^2	• Majority are able to apply the formula: $S_n=\dfrac{n}{2}\left[2a+(n-1)d\right]$
(vi) From (v) The painter can paint 565 m^2 per day. So 2045 m$^2=3\times(565)+350$ m^2 On the 4th day, $S_n=\dfrac{n}{2}\left[2(70)+(n-1)(-3)\right]=350$ $\Rightarrow n(143-3n)=700$ $\Rightarrow -3n^2+143n-700=0$ $\Rightarrow 3n^2-143n+700=0$ $\Rightarrow n=5.5387$ or 42.128 (rejected) Therefore, the painter will use 36 hours to finish painting the house.	• Majority did attempt to find the no. of days required to complete painting by dividing 2045 by 565 • Use direct proportion to find the no. of complete hours which is not acceptable. • Careless in question reading by assuming 1 day as 24 hours which is given that the painter continuously paint for only 10 hours • Careless in question reading, did not leave answer in complete hours