HWA CHONG INSTITUTION

2020 JC2 BLOCK TEST

Higher 2

9758/01

MATHEMATICS

2 July 2020

3 hours

Name:

CT: 1 9

Number of Sheets of Additional Writing Paper Submitted:

Candidates answer on the Question Paper.
Additional Material: List of Formulae 26 (MF26)

READ THESE INSTRUCTIONS FIRST

1. Write your name and class on this Cover Page and any additional writing paper you hand in.
2. Write in dark blue or black pen.
3. You may use an HB pencil for any diagrams or graphs.
4. **Do not use staples,** paper clips, highlighters, glue or **correction fluid.**
5. Answer **all** the questions and write your answers in the spaces provided in the Question Paper.
6. Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
7. Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
8. Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
9. You are reminded of the need for clear presentation in your answers.
10. The number of marks is given in brackets [ ] at the end of each question or part question.

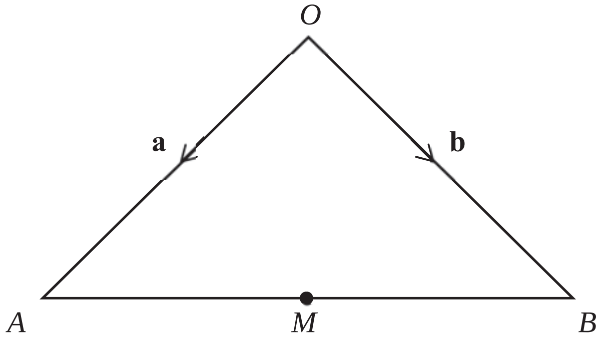
For Examiner's Use

Qn	**Marks**	**Total**	**Remarks**
1		6	
2		7	
3		8	
4		12	
5		13	
6		14	
7		13	
8		13	
9		14	
		100	

This document consists of **29** printed pages and **1** blank page.

Section A: Pure Mathematics [60 marks]

1 **(a)**



The diagram above (not drawn to scale) shows a triangle AOB with $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and M as the mid-point of AB.

Using a suitable scalar product, show that $OM^2 = \frac{1}{4}\left(OA^2 + OB^2 + 2\mathbf{a}\cdot\mathbf{b}\right)$.

Hence find $AM^2 + BM^2 + 2OM^2$ in terms of OA and OB. [4]

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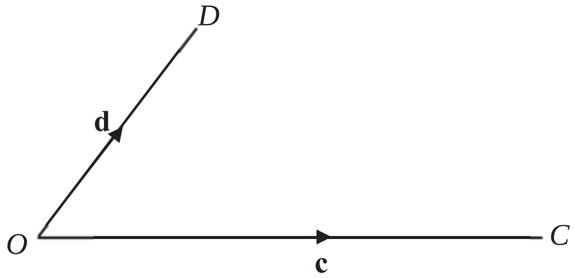
(b)



With reference to the origin O, the points C and D are such that $\overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{OD} = \mathbf{d}$ (see diagram).

(i) Interpret $|\mathbf{c} \times \mathbf{d}|$ geometrically. [1]

(ii) Interpret $|\mathbf{c} \times \hat{\mathbf{d}}|$ geometrically. [1]

2 The curve C has equation

$$y = \frac{x^2 - 7x + 12}{x - 1}.$$

(i) Without using a calculator, find the exact set of values that y can take. [3]

(ii) Draw a sketch of C, which should include stating the equations of any asymptotes and the coordinates of the point(s) where the curve crosses the axes. [2]

(iii) By adding a suitable graph on your sketch of C in part **(ii)**, determine the number of real roots of the equation

$$2x^4 - 14x^3 + 24x^2 + x - 1 = 0 \ .$$ [2]

3 The functions f and g are defined by

$$\mathrm{f}: x \mapsto 1-\frac{2}{\mathrm{e}^{2|x-1|}+1}, \qquad x \in \mathbb{R},$$

$$\mathrm{g}: x \mapsto \frac{3}{x+2}, \qquad x < 1.$$

(i) Explain why the composite function fg exists and find the range of fg. [2]

(ii) Let $\alpha,\ \beta \in \mathbb{R}$. Prove that if $\mathrm{f}\left(\alpha\right) = \mathrm{f}\left(\beta\right)$ and $\alpha \neq \beta$ then $\alpha + \beta = 2$. [2]

(iii) If the domain of f is restricted to $x \le k$, state the greatest value of k for which the function f^{-1} exists. [1]

(iv) Using the domain in part **(iii)**, find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

4 **Do not use a calculator in answering this question.**

(a) Show that $\frac{1}{2}+\frac{1}{2}\text{i}$ is a root of the equation $2z^2-(3-5\text{i})z+4-2\text{i}=0$. Hence, or otherwise, find the other root of the equation in the form $a+b\text{i}$, where $a, b \in \mathbb{R}$. [3]

(b) The complex number *w* is given by $-\frac{3}{\sqrt{2}}+\frac{3}{\sqrt{2}}\text{i}$.

(i) Find the modulus and principal argument of *w*. [2]

(ii) By expressing w^n the form $re^{i\theta}$, where $r > 0$, $n \in \mathbb{Z}$ and $-\pi < \theta \leq \pi$, show that $\frac{w^n - 3^n}{w^n + 3^n}$ can be expressed as $k \tan \frac{3\pi n}{8}$, where k is a complex number to be found. [4]

(iii) State the value of $\frac{2w^n}{w^n+3^n}-\frac{w^n-3^n}{w^n+3^n}$. [1]

(iv) Hence find the first three positive integer values of n for which $\frac{2w^n}{w^n+3^n}$ is purely real. [2]

5 A zoologist has been studying the change in the population of a certain species of squirrel of size n at time t years in each of the 2 islands A and B.

(a) In Island A, he found that the population of the squirrel can be modelled by the differential equation

$$e^n + te^n \frac{dn}{dt} = 4t, \quad \text{where } t \geq 1 .$$

(i) Using the substitution $y = 2te^n$, show that the differential equation can be reduced to $\frac{dy}{dt} = kt$, where k is a positive constant to be determined. [2]

(ii) It is given that when $t = 1$, there are no squirrels on Island *A*. Sketch the solution curve that shows how the population of squirrels on Island *A* changes with respect to time *t* in the context of the question. [4]

(b) On Island B, he found that the annual natural growth rate of the population of the squirrel is 50. However, due to the industrial activities taking place on the island, the squirrels are dying at a rate proportional to number of squirrels present at time t.

(i) It is known that the population will remain constant when there are 5000 squirrels. Find the differential equation relating $\frac{\mathrm{d}n}{\mathrm{d}t}$ and n. [3]

(ii) It is given that there are 2000 squirrels initially on Island *B*. Find n in terms of t. [3]

(iii) Explain what will eventually happen to the population of squirrels on Island *B*. [1]

6 Let $y = \mathrm{e}^{-x}\ln\left(1+3x\right)$. Show that $(1+3x)\left(y+\dfrac{\mathrm{d}y}{\mathrm{d}x}\right) = 3\mathrm{e}^{-x}$. [1]

(i) By further differentiation of the above result, find the Maclaurin series for *y*, up to and including the term in x^3. [4]

(ii) Using expansions from the List of Formulae (MF 26), find a series expansion for y in ascending powers of x, up to and including the term in x^3. Denote this series expansion by m. Comment on the relationship between first three terms of m and the first three terms of the Maclaurin series found in part **(i)**. State a set of values of x for which m converges to y. [5]

(iii) Use your answer in part **(i)**, find an approximate value for $\int_0^2 \mathrm{e}^{-x} \ln(1+3x)\,\mathrm{d}x$. Explain why the approximation is not good. [2]

(iv) Use your answer in part **(i)**, deduce the Maclaurin series for $\mathrm{e}^{x-1} \ln(1-3x)$ in ascending powers of x up to and including the term in x^3. [2]

Section B: Statistics [40 marks]

7 As part of MOE's precautionary measures to prevent the spread of Covid-19 in schools, students are required to record their temperatures daily. Students with a temperature of 38°C or above will be sent home. The temperatures of the C1 and C2 students in a particular school are normally distributed with means and standard deviations as shown in the table below.

	mean (in $^\circ\text{C}$)	standard deviation (in $^\circ\text{C}$)
C1 students	36.5	σ_1
C2 students	36.6	σ_2

(i) The probability of a randomly chosen C1 student having a temperature within 0.6°C from 36.5 is 0.683. Show that $\sigma_1 = 0.6$, correct to 1 decimal place. [2]

Use $\sigma_1 = 0.6$ for the rest of the question.

It is also given that the probability of a randomly chosen C1 student having a higher temperature than a randomly chosen C2 student is 0.449. Find σ_2, correct to 1 decimal place. State one assumption you have made during your calculations. [3]

Use the value of σ_2 correct to 1 decimal place from part **(i)** for the rest of the question.

(ii) A C1 and a C2 student came to school late and were told to take their temperatures at the gate. Given that exactly one of them was sent home, find the probability that it was the C1 student who was sent home. [3]

(iii) Any students who forgot to bring his or her thermometer would be sent to the reception to take his or her temperature. The thermometer at the reception could only read temperatures in Fahrenheit ($^{\circ}\mathrm{F}$). State the distribution of a randomly chosen C2 student's temperature in Fahrenheit. State the parameter(s) of the distribution that you use. Use this distribution to find the probability that a randomly chosen C2 student sent to the reception has a temperature of more than $100^{\circ}\mathrm{F}$. [$\text{Fahrenheit} = \frac{9}{5} \times \text{Celsius} + 32$] [2]

From historical data, the temperatures of randomly chosen staff working in the school follow an unknown distribution with mean $\mu^{\circ}\mathrm{C}$ and standard deviation $1.5^{\circ}\mathrm{C}$.

(iv) Find the probability of drawing a random sample of 50 staff whose mean temperature differs from the true mean by more than $0.5\,^{\circ}\mathrm{C}$ [3]

8 A bag contains 9 numbered balls of identical size. Four of the balls are numbered 3, three of the balls are numbered 4 and two of the balls are numbered 5. In a game, three balls are drawn from the bag at random, without replacement. The random variable S is the sum of the numbers on the three balls drawn.

(i) Show that $\mathrm{P}\left(S=12\right)=\dfrac{25}{84}$ and find the probability distribution of S. [4]

(ii) Show that the probability where the sum of the numbers on the three balls drawn is a multiple of 3 is given by $\frac{29}{84}$. [1]

In a particular game, a player wins \$$a$ if the sum of the numbers on the three balls drawn is a multiple of 3, and nothing otherwise. Let X denote the amount of money won by the player in dollars.

(iii) Given that $\mathrm{E}\left(X\right)=1.45$, show that $a=4.2$ and hence find $\mathrm{Var}\left(X\right)$. [3]

(iv) The player plays the game k times. Let Y denote the number of times the player wins out of k. Find the largest value of k such that the probability that the player wins at least 10 out of k times is less than half. [3]

(v) Suppose the player plays the game 30 times. Find the number of times the player is most likely to win. [2]

9 A group of 8 people are going to watch a movie on a Saturday afternoon. The group consists of 2 single women, 1 single man, 2 married couples and a boy who is the son of a particular married couple.

(i) Find the number of ways in which the group can be seated in a row where at least 2 males are seated together. [2]

(ii) Find the number of ways in which the group can be seated in a row where the boy must be seated between his parents and the other married couple is seated together. [2]

(iii) Find the number of ways in which the group can be seated in a row where the males and females alternate and the family of three are seated together. [3]

Each of the married couple drives a 5-seater car. After the movie, the group decided to go for dinner together at a restaurant which is a 15-minute drive from the cinema.

(iv) Given the married couples are to take their own cars and the boy must follow his parents, find the number of ways where the remaining 3 people in the group can follow the cars to the restaurant. [2]

At the restaurant, the group is given a round table with 10 identical seats.

(v) Find the probability where the family of three must be seated together. [2]

(vi) Given that the family of three must be seated together, find the probability where the 2 single women are separated. [3]

[End of Paper]

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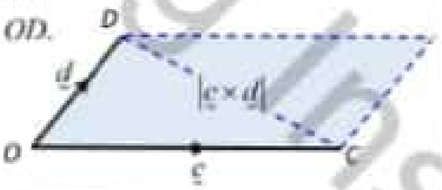
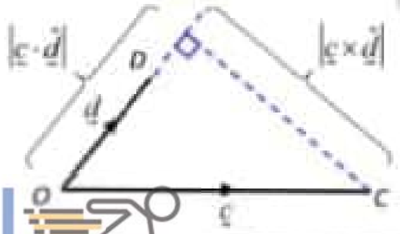
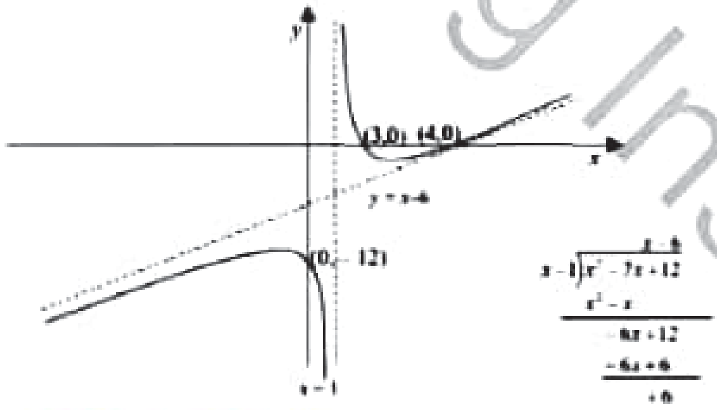
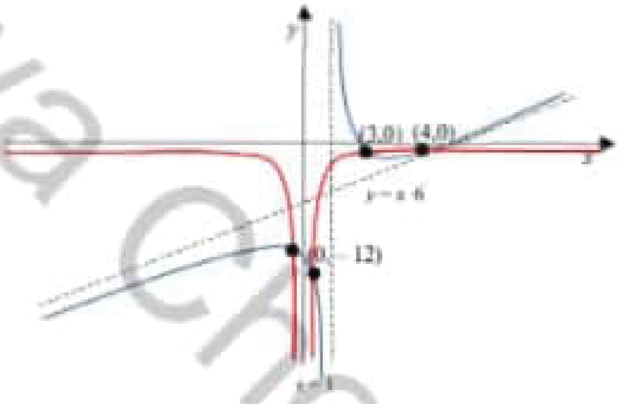
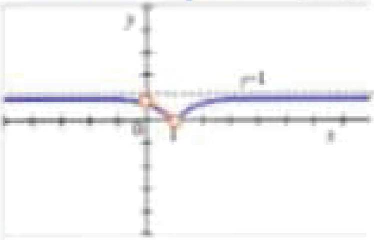
	Suggested Solution	Comments
1(a)	$OM^2=\lvert\overrightarrow{OM}\rvert^2$ $=\overrightarrow{OM}\cdot\overrightarrow{OM}$ $=\frac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b})\cdot\frac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b})$ $=\frac{1}{4}[\underset{\sim}{a}\cdot\underset{\sim}{a}+2\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{b}]$ $=\frac{1}{4}\left[\lvert\underset{\sim}{a}\rvert^2+2\underset{\sim}{a}\cdot\underset{\sim}{b}+\lvert\underset{\sim}{b}\rvert^2\right]$ $=\frac{1}{4}\left(OA^2+OB^2+2\underset{\sim}{a}\cdot\underset{\sim}{b}\right)$ (shown) $\overrightarrow{AM}=\overrightarrow{OM}-\underset{\sim}{a}=\frac{1}{2}(\underset{\sim}{b}-\underset{\sim}{a})$ $\overrightarrow{BM}=\overrightarrow{OM}-\underset{\sim}{b}=\frac{1}{2}(\underset{\sim}{a}-\underset{\sim}{b})$ $\therefore AM^2+BM^2+2OM^2$ $=\overrightarrow{AM}\cdot\overrightarrow{AM}+\overrightarrow{BM}\cdot\overrightarrow{BM}+2OM^2$ $=\frac{1}{2}(\underset{\sim}{b}-\underset{\sim}{a})\cdot\frac{1}{2}(\underset{\sim}{b}-\underset{\sim}{a})+\frac{1}{2}(\underset{\sim}{a}-\underset{\sim}{b})\cdot\frac{1}{2}(\underset{\sim}{a}-\underset{\sim}{b})+\frac{1}{2}\left(OA^2+OB^2+2\underset{\sim}{a}\cdot\underset{\sim}{b}\right)$ $=\frac{1}{4}\left[\lvert\underset{\sim}{b}\rvert^2-2\underset{\sim}{a}\cdot\underset{\sim}{b}+\lvert\underset{\sim}{a}\rvert^2\right]+\frac{1}{4}\left[\lvert\underset{\sim}{a}\rvert^2-2\underset{\sim}{a}\cdot\underset{\sim}{b}+\lvert\underset{\sim}{b}\rvert^2\right]+\frac{1}{2}\left(OA^2+OB^2+2\underset{\sim}{a}\cdot\underset{\sim}{b}\right)$ $=\frac{1}{4}\left[2\lvert\underset{\sim}{a}\rvert^2+2\lvert\underset{\sim}{b}\rvert^2-4\underset{\sim}{a}\cdot\underset{\sim}{b}\right]+\frac{1}{2}\left(OA^2+OB^2+2\underset{\sim}{a}\cdot\underset{\sim}{b}\right)$ $=\frac{1}{2}\left(OA^2+OB^2-\underset{\sim}{a}\cdot\underset{\sim}{b}\right)+\frac{1}{2}\left(OA^2+OB^2+2\underset{\sim}{a}\cdot\underset{\sim}{b}\right)$ $=OA^2+OB^2$	- As required by question, the use of scalar product must be shown explicitly, ie. $OM^2=\overrightarrow{OM}\cdot\overrightarrow{OM}$ $=\frac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b})\cdot\frac{1}{2}(\underset{\sim}{a}+\underset{\sim}{b})$ $=\ldots$ - OM (or equivalently $\lvert\overrightarrow{OM}\rvert$) refers to magnitude or length while $\overrightarrow{OM}$ is a vector, they should not be used interchangeably with one another. Many students misused the notations and were thus not able to make sense of their manipulation. - Vectors must be written as $\underset{\sim}{a}$, $\underset{\sim}{b}$ since it is not possible to write vectors in **bold**.
(b)(i)	$\lvert\underset{\sim}{c}\times\underset{\sim}{d}\rvert$ is area of parallelogram with adjacent sides OC and OD. OR $\lvert\underset{\sim}{c}\times\underset{\sim}{d}\rvert$ is twice the area of ΔOCD. 	
(b)(ii)	$\lvert\underset{\sim}{c}\times\hat{\underset{\sim}{d}}\rvert$ is shortest (or perpendicular) distance from point C to the line passing through O and D. 	- Note that $\lvert\underset{\sim}{c}\cdot\hat{\underset{\sim}{d}}\rvert$ is length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OD}$.
2(i)	$y=\frac{x^2-7x+12}{x-1}$ $yx-y=x^2-7x+12$ $x^2-(7+y)x+(12+y)=0$ Since x is real,	- Note that you are **not allowed** to use calculator for this part of the question. - Another unaccepted method would be to use differentiation to find the y-coordinates of the turning point where one

	Suggested Solution	**Comments**
	$(7+y)^2-4(12+y)\ge 0$ $49+14y+y^2-48-4y\ge 0$ $y^2+10y+1\ge 0$ $(y+5)^2-24\ge 0$ $\lvert y+5\rvert\ge 2\sqrt{6}$ $y\le -5-2\sqrt{6}$ or $y\ge -5+2\sqrt{6}$ **Alternatively,** $y^2+10y+1\ge 0$ Critical values: $y=\dfrac{-10\pm\sqrt{96}}{2}=-5\pm 2\sqrt{6}$  $\{y\in\mathbb{R}: y\le -5-2\sqrt{6}$ or $y\ge -5+2\sqrt{6}\}$	would still need to use the graph (using GC) in one way or another. - Some students use strict inequality (>) which is not correct. Students should use ≥ for real roots. - Read the question carefully and note that answer is to be given in **set notation.** - **Also it is wrong to write $(y+5)^2\ge\pm\sqrt{24}$** - Always make a point to **simplify $\sqrt{24}=2\sqrt{6}$**
2(ii)	 C: $y=\dfrac{x^2-7x+12}{x-1}=x-6+\dfrac{6}{x-1}$	- The x-coordinate of the maximum point is negative so the turning point should be in the 3rd quadrant. - **Show behaviour of curve approaching asymptotes** - **Make sure coordinates** of intercepts are clearly labelled as indicated in the question - Check that long division is done correctly

	Suggested Solution	**Comments**
2(iii)	$2x^4-14x^3+24x^2+x-1=0$ $2x^4-14x^3+24x^2=-(x-1)$ $2x^2(x^2-7x+12)=-(x-1)$ $\frac{x^2-7x+12}{x-1}=-\frac{1}{2x^2}$  A suitable graph is $y=-\frac{1}{2x^2}$. There are **4 points of intersection**, ∴ the no. of root is 4.	- Some students made careless mistakes in the manipulation of the equation. - Some students did not sketch the graph of $y=-\frac{1}{2x^2}$ into the GC and zoom in enough to see the three points of intersection where $x>0$
3(i)	$g:x\mapsto\frac{3}{x+2},\quad x<1$  $R_g=\mathbb{R}\setminus[0,1]\subseteq D_f=\mathbb{R}$ ∴ fg exist. $R_{fg}=(0,1)$ 	- Students are strongly encouraged to sketch the graphs of function f and g on separate diagrams to enable them to find the range of respective functions taking note of each of its domain given in the question - The GC does not show the horizontal asymptote for $y=f(x)$. Students should consider $x\to\pm\infty$ to determine the equation of the horizontal asymptote $y=1$.

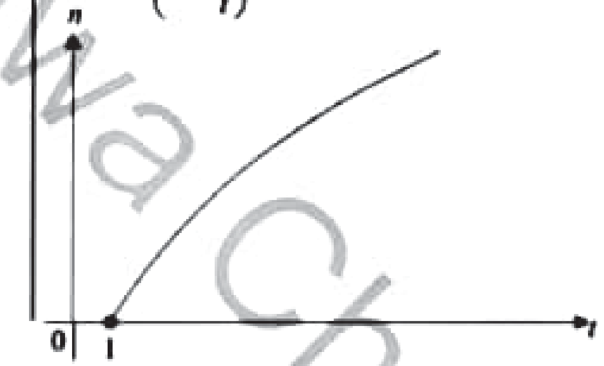
	Suggested Solution	Comments
		• Some students who were able to find R_f made mistakes in set notation, wrongly expressing R_f as $(-\infty,0)$ and $(1,\infty)$.
3(ii)	$f(\alpha)=f(\beta)$ $1-\frac{2}{e^{2\|\alpha-1\|}+1}=1-\frac{2}{e^{2\|\beta-1\|}+1}$ $\|\alpha-1\|=\|\beta-1\|$ $\alpha-1=\beta-1$ or $\alpha-1=-(\beta-1)$ $\alpha=\beta$ (rej $\because \alpha\neq\beta$) $\therefore$ $\alpha+\beta=2$ (shown)	• Students are expected to explain the reason for rejection, especially since working must be showed clearly for a "show" question.
3(iii)	$k=1$	
3(iv)	Let $y=1-\frac{2}{e^{2\|x-1\|}+1}$ $\frac{2}{e^{2\|x-1\|}+1}=1-y$ $e^{2\|x-1\|}=\frac{2}{1-y}-1=\frac{1+y}{1-y}$ $\|x-1\|=\frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$ We note that the restricted domain from part (iii) is $x\leq 1$ $\Rightarrow x-1\leq 0$ $\therefore x-1=-\frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$ $x=1-\frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$ $f^{-1}(x)=1-\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $D_{f^{-1}}=[0,1)$	• Note that f^{-1} exists only when f is a one to one function. • Students are expected to use the restricted domain from part (iii) to find f^{-1}. Since $\|x-1\|=\begin{cases}x-1 & x>1,\\ -(x-1) & x\leq 1.\end{cases}$ Hence $\therefore x-1\neq\frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$

	Suggested Solution	**Comments**
4(a)	Sub $z=\frac{1}{2}+\frac{1}{2}i$ into $2z^2-(3-5i)z+4-2i=0$ LHS: $2\left(\frac{1}{2}+\frac{1}{2}i\right)^2-(3-5i)\left(\frac{1}{2}+\frac{1}{2}i\right)+4-2i$ $=2\left[\frac{1}{4}+\frac{1}{4}i+\frac{1}{4}i-\frac{1}{4}\right]-\left[\frac{3}{2}+\frac{3}{2}i-\frac{5}{2}i+\frac{5}{2}\right]+4-2i$ $=2\left(\frac{1}{2}i\right)-(4-i)+4-2i$ $=0$ Hence, $\frac{1}{2}+\frac{1}{2}i$ is a root of the given equation **Method 1** Sum of roots $=\frac{3-5i}{2}$ $\frac{1}{2}+\frac{1}{2}i+z_2=\frac{3-5i}{2}$ $z_2=\frac{3-5i}{2}-\frac{1}{2}-\frac{1}{2}i=1-3i$ The other root is $1-3i$ **Method 2** $\left(z-\left(\frac{1}{2}+\frac{1}{2}i\right)\right)(2z-k)=2z^2-(3-5i)z+4-2i$ Compare constant term $\left(\frac{1}{2}+\frac{1}{2}i\right)k=4-2i$ $k=\frac{4-2i}{\frac{1}{2}+\frac{1}{2}i}$ $k=\frac{8-4i}{1+i}\times\frac{1-i}{1-i}=\frac{4-12i}{2}=2-6i$	• Some students did not **show working** of how to get 0 after substituting $z=\frac{1}{2}+\frac{1}{2}i$ into the equation. For a question that emphasising no use of calculator, they have to show clear working for each step. • Some students try to solve the quadratic equation using quadratic formula. However, when it comes to finding the square root of a complex number, they either get stuck or did not show any working on how to get the answer (most likely they use G.C., which is not acceptable.). Hence, using quadratic formula is not a good method for this question. • For students who tried factorisation method (method 2), some forgot to have $2z$ (since the coefficient of z^2 is 2) or after they find k, they think that k is the root.

	Suggested Solution	**Comments**
	Since the other factor is $(2z-(2-6i))$ Therefore the other root is $1-3i$	
4(b)(i)	$w=-\frac{3}{\sqrt{2}}+\frac{3}{\sqrt{2}}i$, w lies in the second quadrant $\lvert w\rvert=\sqrt{\left(-\frac{3}{\sqrt{2}}\right)^2+\left(\frac{3}{\sqrt{2}}\right)^2}=3$ Basic Angle $=\tan^{-1}\left(\frac{\frac{3}{\sqrt{2}}}{\frac{3}{\sqrt{2}}}\right)=\tan^{-1}(1)=\frac{\pi}{4}$ $\arg w=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$	- The most common mistake in finding argument is to just use tangent inverse. $\tan^{-1}\left(\frac{\frac{3}{\sqrt{2}}}{-\frac{3}{\sqrt{2}}}\right)$ $=\tan^{-1}(-1)=-\frac{\pi}{4}$ - To find the argument, it is important to draw the Argand diagram to see which quadrant the complex number is in.
4(b)(ii)	$w^n=3^n e^{i\left(\frac{3n\pi}{4}\right)}$ $\frac{w^n-3^n}{w^n+3^n}$ $=\frac{3^n e^{i\left(\frac{3n\pi}{4}\right)}-3^n}{3^n e^{i\left(\frac{3n\pi}{4}\right)}+3^n}$ $=\frac{3^n\left(e^{i\left(\frac{3n\pi}{4}\right)}-1\right)}{3^n\left(e^{i\left(\frac{3n\pi}{4}\right)}+1\right)}$ $=\frac{e^{i\left(\frac{3n\pi}{4}\right)}-1}{e^{i\left(\frac{3n\pi}{4}\right)}+1}$	- Quite a number of students forgot to raise 3 to the power of n when finding w^n and therefore ended up with the wrong answer $3e^{i\left(\frac{3n\pi}{4}\right)}$. This mistake further affects the following part. - Many students were not familiar with the "half angle trick" i.e. applying properties of the conjugates, which is commonly used to evaluate $1\pm e^{i\theta}$. They should revise the "half angle trick". - The alternative method of using trigonometry is

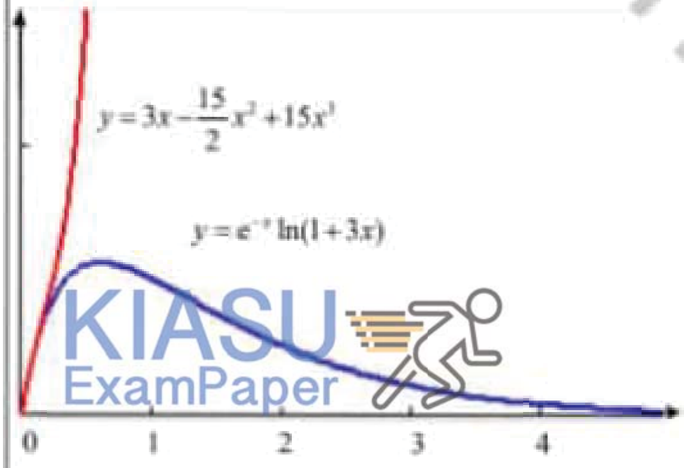
	Suggested Solution	**Comments**
	$=\dfrac{e^{\left(\frac{3\pi n}{8}\right)i}\left(e^{\left(\frac{3\pi n}{8}\right)i}-e^{\left(-\frac{3\pi n}{8}\right)i}\right)}{e^{\left(\frac{3\pi n}{8}\right)i}\left(e^{\left(\frac{3\pi n}{8}\right)i}+e^{\left(-\frac{3\pi n}{8}\right)i}\right)}$ $=\dfrac{2i\sin\left(\frac{3\pi n}{8}\right)}{2\cos\left(\frac{3\pi n}{8}\right)}$ $=i\tan\left(\frac{3\pi n}{8}\right)$ $\therefore k=i$	tedious and not recommended.
4(b)(iii)	$\dfrac{2w^n}{w^n+3^n}-\dfrac{w^n-3^n}{w^n+3^n}=\dfrac{w^n+3^n}{w^n+3^n}=1$	- Many students have computational mistake as following: $\dfrac{2w^n}{w^n+3^n}-\dfrac{w^n-3^n}{w^n+3^n}$ $=\dfrac{2w^n-w^n-3^n}{w^n+3^n}$ $=\dfrac{w^n-3^n}{w^n+3^n}$ - And that is why they cannot get the answer.
4(b)(iv)	$\dfrac{2w^n}{w^n+3^n}$ $=1+\dfrac{w^n-3^n}{w^n+3^n}$ $=1+i\tan\dfrac{3\pi n}{8}$ $=1+i\left(\dfrac{\sin\frac{3\pi n}{8}}{\cos\frac{3\pi n}{8}}\right)$ For $\dfrac{2w^n}{w^n+3^n}$ to be purely real, $\sin\dfrac{3\pi n}{8}=0$ or $\tan\dfrac{3\pi n}{8}=0$ $\dfrac{3\pi n}{8}=k\pi,\ k\in\mathbb{Z}$	- Many students cannot get the previous parts and that is why they cannot even attempt this part. - Some students forgot that n must be integer. - Some students use

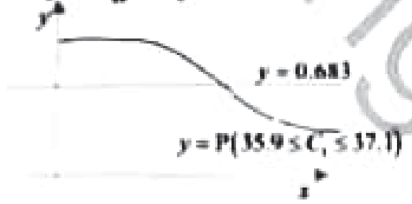
	Suggested Solution	**Comments**
	$\Rightarrow n = \frac{8}{3}k$ Since $k \in \mathbb{Z}$, $k = 3, 6, 9, \ldots$ $\therefore$ the first three positive integer values of $n = 8, 16, 24$	$\arg\left(\frac{2w^n}{w^n + 3^n}\right)$ $= \tan^{-1}\left(\frac{\tan\frac{3\pi n}{8}}{1}\right)$ $= \frac{3\pi n}{8}$ - **They need to be careful with this method because the argument of the complex number** $x + yi$ is NOT simply $\tan^{-1}\frac{y}{x}$. **It depends on which quadrant the complex number is in.**
5(a)(i)	$y = 2te^n$ $\frac{dy}{dt} = 2e^n + 2te^n\frac{dn}{dt}$ Hence $e^n + te^n\frac{dn}{dt} = 4t$ $2e^n + 2te^n\frac{dn}{dt} = 8t$ $\frac{dy}{dt} = 8t$, where $k = 8$.	- Students should use the substitution provided $y = (2t)(e^n)$ and attempt to differentiate implicitly throughout with respect to t together with the use of product rule
5(a)(ii)	$\frac{dy}{dt} = 8t$ $y = 4t^2 + C$ Since $y = 2te^n$, $2te^n = 4t^2 + C$. Since $n = 0$ when $t = 1$, we have $2 = 4 + C \Rightarrow C = -2$. Hence	- Since the DE is to be solved by the substitution method, students should use the result in the previous part to solve for the particular solution y in terms of t first. Thereafter, use the substitution

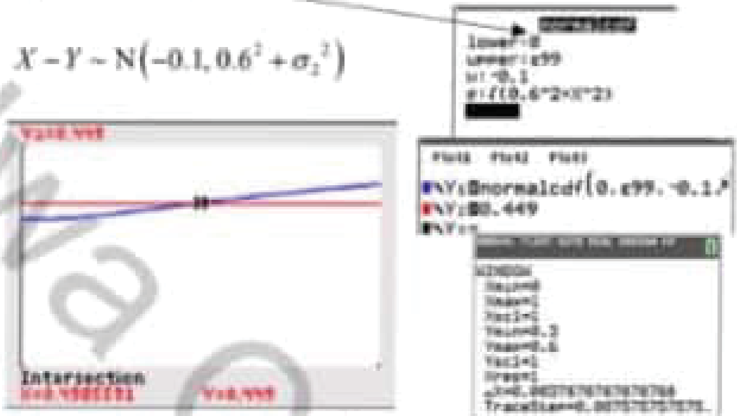
	Suggested Solution	**Comments**
	$e^n = 2t - \frac{1}{t}$ $n = \ln\left(2t - \frac{1}{t}\right)$ $\therefore\ n = \ln\left(2t - \frac{1}{t}\right)$ 	$y = (2t)(e^n)$ to obtain the expression for n in terms of t . • Do note that as t increases, n increases to infinity as well. So there is no horizontal asymptote.
5(b)(i)	$\frac{dn}{dt} = 50 - kn$ Since $\frac{dn}{dt} = 0$ when $n = 5000$, $0 = 50 - 5000k$ $k = 0.01$ $\frac{dn}{dt} = 50 - 0.01n$	• Most students are able to form the first order DE from the question.
5(b)(ii)	$\frac{dt}{dn} = \frac{1}{50 - 0.01n}$ $t = \int \frac{1}{50 - 0.01n}\,dn$ $= \int \frac{100}{5000 - n}\,dn$ $t = -100\ln\|5000 - n\| + C'$ $\ln\|5000 - n\| = -0.01t - C''$ $5000 - n = Ae^{-0.01t}$ where $A = \pm e^{-C''}$ $n = 5000 - Ae^{-0.01t}$ When $t = 0$, $n = 2000$: $A = 3000$	• Many made careless mistake with $t = \int \frac{1}{50 - 0.01n}\,dn$ • Students are required to write out the modulus sign after integration and remove the modulus first to simplify the constant before substituting the intial values to solve for the constant A

	Suggested Solution	**Comments**
	$\therefore\ n = 5000 - 3000e^{-0.01t}$	
5(b)(iii)	As $t \to \infty$, $e^{-0.01t} \to 0$ $\therefore\ n \to 5000$ The population will <u>**increase**</u> and <u>**eventually stabilise at 5000.**</u>	- Students could use the GC to draw a sketch of the graph if they are not sure how to describe the population in context as time tends to infinity. - A handful of students did not state that the population will increase
6	$\frac{dy}{dx} = \left(\frac{3}{1+3x}\right)e^{-x} - e^{-x}\ln(1+3x)$ $= \left(\frac{3}{1+3x}\right)e^{-x} - y$ $\frac{dy}{dx} + y = \left(\frac{3}{1+3x}\right)e^{-x}$ $(1+3x)\left(y + \frac{dy}{dx}\right) = 3e^{-x}$ (shown) **OR** $e^{x}y = \ln(1+3x)$ $e^{x}\frac{dy}{dx} + e^{x}y = \frac{3}{1+3x}$ $e^{x}(1+3x)\left(y + \frac{dy}{dx}\right) = 3$ $(1+3x)\left(y + \frac{dy}{dx}\right) = 3e^{-x}$ (shown)	- Students generally are able to apply **product rule** to obtain $\left(\frac{3}{1+3x}\right)e^{-x} - e^{-x}\ln(1+3x)$ But some failed to identify $y = e^{-x}\ln(1+3x)$ and hence not able to show the required expression. - The **alternative method** requires **product rule and implicit differentiation** followed by factorisation to obtain the required expression
6(i)	$(1+3x)\left(y + \frac{dy}{dx}\right) = 3e^{-x}$ Differentiate throughout wrt x $\left[\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}\right)(1+3x)\right] + 3\left(y + \frac{dy}{dx}\right) = -3e^{-x}$ Differentiate throughout wrt x	- Students should avoid making $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ the subject when performing the repeated differentiations. - Simply apply product rule and implicit

	Suggested Solution	Comments
	$\left[\left(\frac{d^2y}{dx^2}+\frac{d^3y}{dx^3}\right)(1+3x)+\left(\frac{dy}{dx}+\frac{d^2y}{dx^2}\right)(3)\right]+3\left(\frac{dy}{dx}+\frac{d^2y}{dx^2}\right)=3e^{-x}$ At $x=0$, $y=0$, $\frac{dy}{dx}=3$, $\frac{d^2y}{dx^2}=-15$ and $\frac{d^3y}{dx^3}=90$ $\therefore$ The Maclaurin series for y is $0+3x-\frac{15}{2}x^2+\frac{90}{6}x^3+\ldots=3x-\frac{15}{2}x^2+15x^3+\ldots$ (up to x^3 term)	differentiation throughout. Also there is no need to waste time simplifying the expression. • There after students should systematically substitute $x=0$ to first find $\frac{dy}{dx}$, then followed by $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.
6(ii)	$e^{-x}\ln(1+3x)$ $=\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}\right)\left(3x-\frac{(3x)^2}{2}+\frac{(3x)^3}{3}+\ldots\right)$ $=\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}\right)\left(3x-\frac{9x^2}{2}+\frac{27x^3}{3}+\ldots\right)$ $=3x-3x^2+\frac{3x^3}{2}-\frac{9x^2}{2}+\frac{9x^3}{2}+\frac{27x^3}{3}+\ldots$ $\therefore m=3x-\frac{15x^2}{2}+15x^3+\ldots$ The first three terms of y coincide with the first three terms in m.	• Some students did not understand the question. • The intent of the question is for students to use the series expansion provided in MF26 to determine the series for $e^{-x}\ln(1+3x)$ up till the first three terms and denote this series as m. • The first three terms of both series are equal.
	$x\in\mathbb{R}$ and $-1<3x\le1$ $\left\{x : x\in\mathbb{R}, -\frac{1}{3}<x\le\frac{1}{3}\right\}$	• For the set of values of x where m converges to y , students should note that the series expansion for $e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots$ is valid for all $x\in R$ while the the series expansion for

	Suggested Solution	Comments
		$\ln(1+3x)=3x-\frac{(3x)^2}{2}+\frac{(3x)^3}{3}+\ldots$ is valid when $-1<3x\le 1$ • Note that set notation is required in the question
6(iii)	$\int_0^2 e^{-x}\ln(1+3x)\,\mathrm{d}x$ $\approx\int_0^2\left[3x-\frac{15}{2}x^2+15x^3\right]\mathrm{d}x$ $=\left[\frac{3x^2}{2}-\frac{15x^3}{6}+\frac{15x^4}{4}\right]_0^2$ $=6-20+60$ $=46$ Since $[0,2]\not\subset\left(-\frac{1}{3},\frac{1}{3}\right]$ (i.e. range of validity) of the series expansion, it will not be a good approximation.  As shown in the above diagram, the area below both curves will be different when the limits of integration ranges from 0 to 2. Hence it will not be a good approximation.	• Using the first three terms of the series expansion of $e^{-x}\ln(1+3x)$, the approximate $\int_0^2 e^{-x}\ln(1+3x)\,\mathrm{d}x$ could be obtained either by integration or through the use of GC. Some students forgot to integrate the series expansion. • The series expansion is valid for values of $x=\left(-\frac{1}{3},\frac{1}{3}\right]$. Students are required to explain that $[0,2]$ does not lie within the range of validity where the approximations would be valid. • Alternatively, a clearly labelled graph could be drawn to explain why the approximation is not good • Some students use the GC to compare the actual area of $\int_0^2 e^{-x}\ln(1+3x)\,\mathrm{d}x$ with the approximate area found

	Suggested Solution	**Comments**
6(iv)	$\mathrm{f}(x)=\mathrm{e}^{x-1}\ln(1-3x)$ $=\frac{1}{\mathrm{e}}\{\mathrm{e}^{-(-x)}\ln[1+3(-x)]\}$ <<Replace x with $-x$>> $=-\frac{3x}{\mathrm{e}}-\frac{15x^2}{2\mathrm{e}}-\frac{15x^3}{\mathrm{e}}+\ldots$ <<Using the result in part (i)>>	- It is wrong to replace x with $1-x$ - Students should first express e^{x-1} as $(\mathrm{e}^x)(\mathrm{e}^{-1})$ $\mathrm{f}(x)=\mathrm{e}^{x-1}\ln(1-3x)$ $\mathrm{f}(x)=\mathrm{e}^{-1}[\mathrm{e}^x\ln(1-3x)]$ $\mathrm{f}(x)=\mathrm{e}^{-1}[\mathrm{e}^{-(-x)}\ln(1+3(-x))]$
7(i)	Let X and Y be the temperatures of randomly chosen C1 and C2 students respectively (in °C). $X\sim \mathrm{N}(36.5,\sigma_1^2)$ and $Y\sim \mathrm{N}(36.6,\sigma_2^2)$ $\mathrm{P}(\lvert X-36.5\rvert\le 0.6)=0.683$ $\mathrm{P}(35.9\le X\le 37.1)=0.683$ **Method 1: (Using GC)**  $\sigma_1=0.5996151\approx 0.6$ (1 d.p) (shown) **Method 2: (Standardise and using InvNorm)** $\mathrm{P}\left(\left\lvert\frac{X-36.5}{\sigma_1}\right\rvert\le\frac{0.6}{\sigma_1}\right)=0.683$ $\mathrm{P}\left(-\frac{0.6}{\sigma_1}\le Z\le\frac{0.6}{\sigma_1}\right)=0.683$ Using InvNorm, $-\frac{0.6}{\sigma_1}=-1.0006$, $\therefore\sigma_1=0.6$ (to 1d.p)	- Students are required to **define the random variables clearly** and state the required probability. For the required probability, do **note the inequality sign because of the word "within"**. Thereafter they are required to provide a sketch of the graphs of $y=\mathrm{P}(35.9\le C_1\le 37.1)$ and $y=0.683$ and use the GC to obtain the point of intersection so as to obtain full credit. - Alternatively, students could solve by standardising the random variable and use invnorm to solve for σ_1

	Suggested Solution	Comments
	Method 1: (Using GC) $P(X>Y)=P(X-Y>0)=0.449$ $X-Y\sim N\left(-0.1,0.6^2+\sigma_2^2\right)$ lower:0 upper:ε99 μ:-0.1 σ:√(0.6^2+X^2) Y1=normalcdf(0,ε99,-0.1,√ Y2=0.449 WINDOW Xmin=0 Xmax=1 Xscl=1 Ymin=0.2 Ymax=0.6 Yscl=1 Xres=1 ΔX=0.0037878787878788 TraceStep=0.0075757575757575 Intersection X=0.4985591 Y=0.449 $\sigma_2=0.4985591\approx 0.5$ (1 d.p) **Method 2: (Standardise and using InvNorm)** $P\left(\frac{X-Y-(-0.1)}{\sqrt{\sigma_1^2+\sigma_2^2}}>\frac{0-(-0.1)}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)=0.449$ $P\left(Z>\frac{0.1}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)=0.449$ $P\left(Z<\frac{0.1}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)=0.551$ using InvNorm, $\frac{0.1}{\sqrt{\sigma_1^2+\sigma_2^2}}=0.12819$ (to 5 s.f.) $\frac{0.1}{\sqrt{0.6^2+\sigma_2^2}}=0.12819$ $\sigma_2^2=\left(\frac{0.1}{0.12819}\right)^2-0.6^2$ $\sigma_2=0.49854=0.5$ (to 1 d.p.) **Assumption:** The **temperature** of any randomly chosen **C1 student** and the **temperature** of any randomly chosen **C2 student** are **independent** of each other.	- For the second part, take note that the new random variable could be $X-Y$ or $Y-X$ depending on the inequality form $P(X>Y)$ $=P(X-Y>0)$ $=P(Y-X<0)$ $X-Y\sim N\left(-0.1,0.6^2+\sigma_2^2\right)$ $Y-X\sim N\left(0.1,0.6^2+\sigma_2^2\right)$ - For both cases, the variance is the same but care has to be taken when keying in the standard deviation into GC as $\sqrt{0.6^2+X^2}\neq 0.6+X$ - Similar algebraic manipulation error such as $\sqrt{0.6^2+X^2}\neq 0.6+X$ is also made for some of those who solve the question by standardising the random variable - Common mistake for assumption: X and Y are normally distributed. This is wrong because it is already a condition given in the question.

	Suggested Solution	**Comments**
7(ii)	$\text{P}(\text{C1 student sent home} \mid \text{only 1 is sent home})$ $= \dfrac{\text{P}(\text{C1 student sent home} \cap \text{C2 student not sent home})}{\text{P}(\text{only 1 is sent home})}$ $= \dfrac{\text{P}(X \geq 38.0 \cap Y < 38.0)}{\text{P}(X \geq 38.0 \cap Y < 38.0) + \text{P}(X < 38.0 \cap Y \geq 38.0)}$ $= \dfrac{\text{P}(X \geq 38.0) \times \text{P}(Y < 38.0)}{\text{P}(X \geq 38.0) \times \text{P}(Y < 38.0) + \text{P}(X < 38.0) \times \text{P}(Y \geq 38.0)}$ $= \dfrac{0.0061938129}{0.0061938129 + 0.0025393237}$ $= 0.709$ (to 3s.f.)	- Students are to note that they are required to compute a conditional probability. - $\text{P}(\text{only 1 is sent home})$ means either "C1 sent home and C2 remained in school" or "C2 sent home and C1 remained in school", therefore it is an addition of two events.
7(iii)	Let F be the temperature of a randomly chosen C2 student (°F). $F = \dfrac{9}{5}Y + 32 \sim \text{N}(97.88, 0.81)$ $\text{P}(F > 100) = 0.00925$ (to 3 s.f.)	- For those who did not score for this part mainly made errors in computing the variance for the new random variable. - $\text{Var}(F) = \left(\dfrac{9}{5}\right)^2 \text{Var}(Y)$ - Note that variance of a constant is zero. i.e. $\text{Var}(32) = 0$
7(iv)	Let $\bar{W}$ be the sample mean temperature of 50 randomly chosen staff. Since $n = 50$ is large, **by Central Limit Theorem**, $\bar{W} \sim \text{N}\left(\mu, \dfrac{1.5^2}{50}\right)$ **approximately**. $\text{P}\left(\left\lvert \bar{W} - \mu \right\rvert > 0.5\right)$ $= \text{P}\left(\left\lvert \dfrac{\bar{W} - \mu}{\frac{\sigma}{\sqrt{50}}} \right\rvert > \dfrac{0.5}{\frac{\sigma}{\sqrt{50}}}\right)$ $= \text{P}\left(\lvert Z \rvert > \dfrac{0.5\left(\sqrt{50}\right)}{1.5}\right)$ $= \text{P}(Z > 2.357022604) + \text{P}(Z < -2.357022604)$	- When quoting use of Central Limit Theorem to approximate the distribution of Sample Mean Temperature, students are required to state clearly the words in bold. - Most students solve the question by standardising the random variable while a handful also consider the distribution of $\bar{W} - \mu \sim \text{N}\left(0, \dfrac{1.5^2}{50}\right)$

	Suggested Solution	Comments
	=0.0184220923 =0.0184 (3 s.f.)	• Students should note the following: $P(\|Z\|>k)$, where k is a positive constant $=P(Z>k)+P(Z<-k)$ $P(\|Z\|<k)$, where k is a positive constant $=P(-k<Z<k)$ And $Z\sim N(0,1)$
8(i)	When listing out all the outcomes, do it systematically: (a) All 3 balls have the same number 3+3+3 = 9 4+4+4 = 12 5+5+5 = 15 (b) Only 2 balls have the same number 3+3+(4 or 5) = 10 or 11 4+4+(3 or 5) = 11 or 13 5+5+(3 or 4) = 13 or 14 (c) All 3 balls different numbers 3+4+5 = 12 $P(S=12)=\frac{{}^4C_1\times{}^3C_1\times{}^2C_1}{{}^9C_3}+\frac{{}^3C_3}{{}^9C_3}=\frac{25}{84}$ Or $P(S=12)=\left(\frac{4}{9}\times\frac{3}{8}\times\frac{2}{7}\times3!\right)+\left(\frac{3}{9}\times\frac{2}{8}\times\frac{1}{7}\right)=\frac{25}{84}$ (shown) $P(S=9)=\frac{{}^4C_3}{{}^9C_3}=\frac{1}{21}$ Or $P(S=9)=\frac{4}{9}\times\frac{3}{8}\times\frac{2}{7}=\frac{1}{21}$ $P(S=10)=\frac{{}^4C_2\times{}^3C_1}{{}^9C_3}=\frac{3}{14}$ Or $P(S=10)=\frac{4}{9}\times\frac{3}{8}\times\frac{3}{7}\times3=\frac{3}{14}$	• Most students could prove list out the possible outcomes to get $S=12$ and obtain the answer $\frac{25}{84}$. A handful of students solved by direct probability but fail to consider the permutation of the outcome {3,4,5}. • Several students did not verify that $\sum P(S=s)=1$ and end up with an incorrect probability distribution table due to some errors in calculations.

<table>
<tr><th></th><th>Suggested Solution</th><th>Comments</th></tr>
<tr><td></td><td>$\mathrm{P}(S=11)=\frac{{}^4C_2\times{}^2C_1}{{}^9C_3}+\frac{{}^4C_1\times{}^3C_2}{{}^9C_3}=\frac{2}{7}$
<table><tr><td>s</td><td>9</td><td>10</td><td>11</td><td>12</td><td>13</td><td>14</td></tr><tr><td>$\mathrm{P}(S=s)$</td><td>$\frac{1}{21}$</td><td>$\frac{3}{14}$</td><td>$\frac{2}{7}$</td><td>$\frac{25}{84}$</td><td>$\frac{5}{42}$</td><td>$\frac{1}{28}$</td></tr></table>
Or $\mathrm{P}(S=11)=\left(\frac{4}{9}\times\frac{3}{8}\times\frac{2}{7}\times3\right)+\left(\frac{3}{9}\times\frac{2}{8}\times\frac{4}{7}\times3\right)=\frac{2}{7}$
$\mathrm{P}(S=13)=\frac{{}^4C_1\times{}^2C_2}{{}^9C_3}+\frac{{}^3C_2\times{}^2C_1}{{}^9C_3}=\frac{5}{42}$
Or $\mathrm{P}(S=13)=\left(\frac{3}{9}\times\frac{2}{8}\times\frac{2}{7}\times3\right)+\left(\frac{2}{9}\times\frac{1}{8}\times\frac{4}{7}\times3\right)=\frac{5}{42}$
$\mathrm{P}(S=14)=\frac{{}^3C_1\times{}^2C_2}{{}^9C_3}=\frac{1}{28}$
Or $\mathrm{P}(S=14)=\frac{2}{9}\times\frac{1}{8}\times\frac{3}{7}\times3=\frac{1}{28}$</td><td></td></tr>
<tr><td>8(ii)</td><td>Required probability $=\mathrm{P}(S=9)+\mathrm{P}(S=12)$
$=\frac{1}{21}+\frac{25}{84}=\frac{29}{84}$</td><td></td></tr>
<tr><td>8(iii)</td><td>$\mathrm{E}(X)=0\mathrm{P}(X=0)+a\mathrm{P}(X=a)$
$=0+\frac{29}{84}a=1.45$
$\therefore a=4.2$
$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$
$=\left[0^2\mathrm{P}(X=0)+4.2^2\mathrm{P}(X=a)\right]-(1.45)^2$
$=4.2^2\left(\frac{29}{84}\right)-1.45^2=3.9875=\frac{319}{80}$</td><td>• Most students are able to prove that $a=4.2$.
• Some students assumed that X is a binomial random variable. This is not correct. (The outcomes of X is 0 or 4.2, while the outcomes required for a single independent trial of binomially distributed R.V is 0 and 1.)
• A minority of students rounded off 3.9875 to 3.99. 3.9875 is an exact value not to be rounded off. Non-exact values</td></tr>
</table>

<table>
<tr><th></th><th>Suggested Solution</th><th>Comments</th></tr>
<tr><td></td><td></td><td>that should be rounded off include rational numbers or numbers with recurring decimals.</td></tr>
<tr><td>8(iv)</td><td>

$Y \sim \mathrm{B}\left(k, \frac{29}{84}\right)$

$\mathrm{P}(Y \geq 10) < 0.5$
$1 - \mathrm{P}(Y \leq 9) < 0.5$
$\mathrm{P}(Y \leq 9) > 0.5$

Using G.C.

<table>
<tr><th>k</th><th>$\mathrm{P}(Y \leq 9)$</th></tr>
<tr><td>27</td><td>0.53711</td></tr>
<tr><td>28</td><td>0.48196</td></tr>
</table>

Therefore, largest value of $k = 27$.

</td><td>

- The intention of the question is to use the GC binocdf function to solve the inequality involving k.
- Some students took the complement wrongly, including $Y = 10$ in the complement and therefore attaining an erroneous answer.

</td></tr>
<tr><td>8(v)</td><td>

$Y \sim \mathrm{B}\left(30, \frac{29}{84}\right)$

Using G.C.

<table>
<tr><th>y</th><th>$\mathrm{P}(Y = y)$</th></tr>
<tr><td>9</td><td>0.13689</td></tr>
<tr><td>10</td><td>0.15158</td></tr>
<tr><td>11</td><td>0.14531</td></tr>
</table>

Since $\mathrm{P}(Y = 10)$ has the highest probability, the player is most likely to win 10 out of the 30 games.

</td><td>

- The most common mistake students make upon attempting this question is to find the mean and rounding to the nearest integer.
- Take note that the "number of games most likely to win" is referring to the mode of the distribution, i.e. $Y = y$ with the highest probability.

</td></tr>
<tr><td>9(i)</td><td>

Method 1 : By Complement
Number of ways without restriction = 8!
Number of ways in which all the 4 males are separated

$= \binom{5}{4} \times 4! \times 4!$

Number of ways in which at least 2 males are seated together in a row $= 8! - \binom{5}{4} \times 4! \times 4! = 37440$

</td><td>

- The opposite of "at least 2 males together" is "all males are separate".

Method 1 is shorter and is preferred.

</td></tr>
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	Suggested Solution	Comments
	Method 2 : By Listing $\uparrow\boxed{F_1}\uparrow\boxed{F_2}\uparrow\boxed{F_3}\uparrow\boxed{F_4}\uparrow$ **Case 1:** $\boxed{\text{2 Males in a group}}\ \boxed{M}\ \boxed{M}$ 5 slots for the 2 Males in a group $\binom{4}{2}$ ways to slot the other 2 Males Number of ways $= 5\times\binom{4}{2}\times4!\times4!$ OR $5\times\binom{4}{2}\times4!\times\binom{4}{2}2!2!$ $=17280$ **Case 2:** $\boxed{\text{2 Males in a group}}\ \boxed{\text{2 Males in a group}}$ $\binom{5}{2}$ ways to slot the 2 groups of 2 Males in a group Number of ways $=\binom{5}{2}\times4!\times4!=5760$ **Case 3:** $\boxed{\text{3 Males in a group}}\ \boxed{M}$ 5 slots for the 3Males in a group 4 slots for the other Male Number of ways $= 5\times4\times4!\times4!=11520$ **Case 4** : All 4 Males in a group Number of ways $= 5\times4!\times4!= 2880$	- Be sure to list out all cases if using this method. - Some students who used Method 2 considered 2 males together, but did not take into account that the other 2 males can be either separate or together.
9(ii)	$\boxed{FBMo}\boxed{W}\boxed{W}\boxed{M}\boxed{MW}$ Required number of ways $=5!\times2!\times2!$ $=480$	- Consider the family of 3 as 1 item. Consider the other couple as 1 item. - No. of ways to arrange 5 items = 5!. - The father and mother can arrange themselves in 2! ways. Likewise for the other couple.

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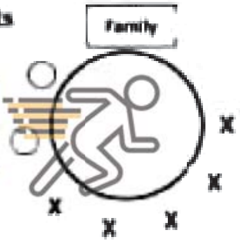
9(iii)	1) $\boxed{FMoB}\boxed{W}\boxed{M}\boxed{W}\boxed{M}\boxed{W}$ 2) $\boxed{W}\boxed{FMoB}\boxed{W}\boxed{M}\boxed{W}\boxed{M}$ 3) $\boxed{W}\boxed{M}\boxed{W}\boxed{FMoB}\boxed{W}\boxed{M}$ 4) $\boxed{W}\boxed{M}\boxed{W}\boxed{M}\boxed{W}\boxed{FMoB}$ 5) $\boxed{M}\boxed{W}\boxed{M}\boxed{W}\boxed{FMoB}\boxed{W}$ 6) $\boxed{M}\boxed{W}\boxed{FMoB}\boxed{W}\boxed{M}\boxed{W}$ Required number of ways $= 6\times 3!\ \times 2!\ \times 2! = 144$	- Since male and female alternate, the mother must be between her husband and her son. - Now there are 6 cases where males and females alternate. Within each case, the father and son can be arranged in 2! ways, the other 3 women in 3! ways, and the other 2 men in 2! ways.
9(iv)	Car A: FMoB,X,X Car B: CC,X,X,X Case 1: All three go into Car B. Case 2: Exactly two of the singles go into Car B. Case 3: Exactly one of the singles goes into Car B. Required number of ways $= 1 + {}^{3}C_{2} + {}^{3}C_{1}$ $= 7$	- Either 3 people, 2 people or 1 person can go into the same car as the couple. So there are 3 cases to consider.
9(v)	**Method 1:** 8 people with 2 empty seats  Required number of ways $= \dfrac{\dfrac{8!}{8(2!)}\times 3!}{\dfrac{10!}{10(2!)}}$ $= \dfrac{1}{12}$ (or 0.0833)	- Consider the family of 3 as 1 item. No. of ways to arrange 6 items and 2 empty seats $= \dfrac{8!}{8(2!)}$. - The family of 3 can be arranged among themselves in 3! ways. - No. of ways to arrange 8 people and 2 empty

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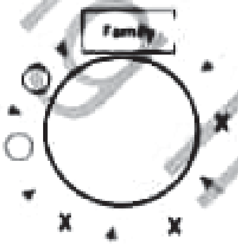
		seats with no restrictions = $\frac{10!}{10(2!)}$.
	Method 2: Let the boy sit down first. **Case 1: Father sits next to the boy** P(father is next to boy) = $\frac{2}{9}$ P(mother is next to father & boy) = $\frac{2}{8}$ **Case 2: Father is not next to the boy** P(father is 1 seat away from boy) = $\frac{2}{9}$ P(mother is between father & boy) = $\frac{1}{8}$ P(family is together) = $\frac{2}{9}\frac{2}{8}+\frac{2}{9}\frac{1}{8}=\frac{1}{12}$	
9(vi)	 **Method 1:** $\text{P}(\text{2 single women separated}\mid\text{family together})$ $=\frac{\text{P}(\text{2 single women separated}\cap\text{family together})}{P(\text{family together})}$ $=\frac{\frac{6!}{6(2!)}\times 3!\times {}^6C_2\times 2!}{\frac{8!}{8(2!)}\times 3!}$ $=\frac{5}{7}$ (or 0.714)	- It is given that the family of 3 must be seated together. So this is a conditional probability. Some students did not realise this and only found P(2 single women separate and family of 3 together). - Consider the family of 3 as 1 item. Ignoring the 2 single women, no. of ways to arrange the family, the other 3 people and 2 empty seats = $\frac{(6-1)!}{2!}3!$. - There are 6 slots to slot in the 2 single women. No. of ways to slot in

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		the 2 single women = ${}^6C_2 \times 2!$. - Then we divide this by the no. of ways where the family of 3 are together.
	Method 2 (complementation): $P(\text{2 single women separated}\mid\text{family together})$ $=1-P(\text{2 single women together}\mid\text{family together})$ $=1-\dfrac{\dfrac{7!\times 2!\times 3!}{7\times 2!}}{\dfrac{8!}{8(2!)}\times 3!}$	
	Method 3 (reduced sample space): Let the family of 3 sit together first. No. of ways of arranging the 3 other people and 2 empty seats = $\dfrac{5!}{2!}$ Slot in the 2 single women in ${}^6C_2\ 2!$ ways P(2 single women separate $\mid$ family together) $=\dfrac{\dfrac{5!}{2!}\,{}^6C_2\,2!}{\dfrac{7!}{2!}}=\dfrac{5}{7}$	
	Method 4: Let the family of 3 sit together first. **Case 1: The 1st single woman is next to the family** P(1st single woman is next to family) = $\dfrac{2}{7}$ P(2nd single woman is separate from 1st woman) = $\dfrac{5}{6}$ **Case 2: The 1st single woman is not next to family** P(1st single woman is not next to family) = $\dfrac{5}{7}$ P(2nd single woman is separate from 1st woman) = $\dfrac{4}{6}$ P(2 single women separate $\mid$ family together) $=\dfrac{2}{7}\dfrac{5}{6}+\dfrac{5}{7}\dfrac{4}{6}=\dfrac{5}{7}$	