1 Question 1 (TMJC 2020 Promos Q8)

The plane π contains the point A with coordinates $(-2,1,4)$ and the line with equation $\mathbf{r}=\begin{pmatrix}0\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}-2\\0\\1\end{pmatrix}$, where λ is a parameter.

(i) Show that the cartesian equation of π is $x+y+2z=7$. [2]

Let L be the set of lines such that the equation of any line l_a in L is given by $\mathbf{r}=\begin{pmatrix}-2a-2\\1\\4a+2\end{pmatrix}+\mu\begin{pmatrix}-4-a\\1\\2a+3\end{pmatrix}$, where μ is a parameter and a is a real constant.

(ii) Verify that the point B with coordinates $(6,-1,-4)$ lies on any line l_a in L. [2]

The equation of a particular line l_c in L is given by $\mathbf{r}=\begin{pmatrix}-2c-2\\1\\4c+2\end{pmatrix}+\mu\begin{pmatrix}-4-c\\1\\2c+3\end{pmatrix}$, where μ is a parameter and c is a real constant. It is given that l_c is parallel to π.

(iii) Show that $c=-1$. [1]

(iv) Find the exact distance between l_c and π. [2]

(v) Point F is the foot of perpendicular from $B(6,-1,-4)$ to π and point C is on l_c. Given that the area of triangle BCF is $\frac{\sqrt{66}}{4}$ units^2, find possible position vectors of C. [5]

2 Question 2 (RI 2020 Promos Q2)

2 Referred to an origin O, points A and B have position vectors given respectively by $\mathbf{i}+4\mathbf{j}+8\mathbf{k}$ and $6\mathbf{i}+2\mathbf{j}+3\mathbf{k}$. The point P on AB is such that $AP:PB=1-\lambda:\lambda$.

(i) Show that the exact area of triangle OAB is $\dfrac{5}{2}\sqrt{101}$. [1]

(ii) Find the exact value of λ for which OP is perpendicular to AB. [3]

(iii) Find the exact value of λ for which angles AOP and POB are equal. [3]

3 Question 3 (TJC 2020 Promos Q9)

The points A, B and C lie on the circumference of a circle with center O and diameter AC. It is given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(i) Find $\overrightarrow{BC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Hence show that AB is perpendicular to BC. [3]

The point D is on BC such that $BD : DC = 3 : 2$.

(ii) Show that the area of triangle OCD can be written as $k|\mathbf{a}\times\mathbf{b}|$ where k is a real constant to be found. [3]

(iii) Given that angle $AOB = 120^o$, find $\overrightarrow{ON}$ in terms of $\mathbf{a}$ where N is the foot of perpendicular of D to AC. [4]

4 Question 4 (TJC 2020 Promos Q10)

Points A and B have coordinates $(1,2,1)$ and $(1,-5,2)$ respectively. The line BC has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

(i) Given that $AB = AC$, show that the coordinates of point C is $(-3,-3,4)$. [3]

(ii) Find the cosine of angle BAC and hence find the exact area of triangle ABC. [4]

(iii) Plane p is parallel to plane ABC and it is $\sqrt{\frac{11}{6}}$ units from plane ABC. Find the possible equations of p. [4]

(iv) Point D lies on p. Deduce the exact volume of the tetrahedron $ABCD$. [2]

[The volume of a tetrahedron of base area A and height h is given by $V = \frac{1}{3}Ah$.]

5 Question 5 (RI 2020 Promos Q9)

The plane p passes through the points with coordinates $(2, 2, 4)$, $(0, 6, 8)$ and $(-2, -2, -3)$.

(i) Find a cartesian equation of p. [4]

The line l has equation $x-4=4-y,\ z=9$.

(ii) Show that the coordinates of the point of intersection of l and p is $(2, 6, 9)$. [2]

(iii) Find an equation of the line which is a reflection of l in p. [6]

6 Question 6 (NJC 2020 Promos Q8)

Points A, B and C are non-collinear and origin O is not on the plane ABC. Point P has position vector given by $\overrightarrow{OP} = \lambda\overrightarrow{OA} + \mu\overrightarrow{OB} + (1-\lambda-\mu)\overrightarrow{OC}$, where $\lambda, \mu \in \mathbb{R}$.

(i) Prove that P lies in the plane ABC. [2]

It is now given that $\overrightarrow{OA} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\overrightarrow{OB} = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{OC} = -2\mathbf{i} + \mathbf{k}$. Furthermore, P is the point of reflection of B in the line passing through A and C.

(ii) Find the value of λ and of μ. [5]

7 Question 7 (NJC 2020 Promos Q7)

Referred to the origin, points A and B have non-zero position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively.

(i) Show that $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=-2(\mathbf{a}\times\mathbf{b})$. [2]

It is given that $|\mathbf{b}|=2|\mathbf{a}|$.

(ii) Find the maximum value of $|(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})|$ in terms of $|\mathbf{a}|$. [3]

It is further given that A is a fixed point on the x-axis and B is a variable point on the xz-plane.

(iii) Given that the value of $|(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})|$ is maximum, write down all possible expressions for $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})$ in terms of $|\mathbf{a}|$. [2]

8 Question 8 (EJC 2020 Promos Q12)

Stereophotogrammetry is a method of determining coordinates of points in the three-dimensional (3-D) replication of physical scenes. It relies on using multiple images taken by digital cameras from different positions.

In a simplistic model for this process, the camera sensors are represented by planes with finite size. A *ray of sight* for a particular point in the physical scene is defined as the line passing through its image point on the camera sensor and the focal point of the camera. The 3-D coordinates of a particular point is the intersection of the rays of sight from the different cameras.

For a particular set up, Camera 1 has focal point, F_1 at $(20, 5, 10)$ and Camera 2 has focal point, F_2 at $(-20, 5, 10)$. The image point of a point *A*, the highest tip of a flag pole, on Camera 1 is $A_1\left(\frac{61}{3}, 0, \frac{28}{3}\right)$, and the image point of *A* on Camera 2 is $A_2\left(-\frac{67}{3}, 0, \frac{28}{3}\right)$.

(i) Find the vector equations of l_1 and l_2, the rays of sight for point *A* from the Cameras 1 and 2 respectively, and hence find the coordinates of point *A*. [4]

The base of the flag pole is known to be on the plane *P* that contains the point $D(75, 90, -50)$ and the line *L* with equation $\mathbf{r} = \begin{pmatrix} 23 \\ 90 \\ -46 \end{pmatrix} + s\begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix}, s \in \mathbb{R}$.

(ii) Show that the vector equation of the plane *P* is $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 13 \end{pmatrix} = -755$. [3]

The flag pole was erected perpendicular to the base plane *P*.

(iii) Find the coordinates of the point *B*, the base of the flag pole. [3]

(iv) Hence or otherwise, find the length of the flag pole. [2]

(v) Given that the horizontal plane in this model is the *x*-*y* plane, find the angle of incline for the plane *P* from the horizontal plane. [2]

9 Question 9 (EJC 2020 Promos Q5)

(a) The variable vector $\mathbf{u}$ satisfies the following equations:

$$\mathbf{u} \cdot \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = -6 \text{ and}$$

$$\mathbf{u} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = k \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \text{ for } k \in \mathbb{R}, k \neq 0.$$

(i) Explain why $\mathbf{u} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$. [1]

(ii) Hence or otherwise, find the set of vectors $\mathbf{u}$ and describe this set geometrically. [3]

(b) The points A, B and C have distinct non-zero position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively and the vectors satisfy the equation $\mathbf{c} = \lambda \mathbf{a} + (1-\lambda)\mathbf{b}$ where $\lambda \in \mathbb{R}, \lambda \neq 0, \lambda \neq 1$. Prove that the points A, B and C are collinear. [2]

(c) Given that the point P has a non-zero position vector $\mathbf{p}$ and that the plane Π has equation $\mathbf{r} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is a unit vector, state the geometrical meaning of $|\mathbf{p} \cdot \mathbf{n}|$ in relation to the point P and the plane Π. [1]

10 Answers

1. (iv) $\frac{5\sqrt{6}}{3}$ units.

   (v) $\overrightarrow{OC} = \frac{1}{10}\begin{pmatrix} 51 \\ -7 \\ -37 \end{pmatrix}$ or $\frac{1}{10}\begin{pmatrix} 69 \\ -13 \\ -43 \end{pmatrix}$.

2. (ii) $\lambda = \frac{11}{54}$.

   (iii) $\lambda = \frac{7}{16}$.

3. (i) $\overrightarrow{BC} = -\mathbf{a} - \mathbf{b}$.

   (ii) $k = \frac{1}{5}$.

   (iii) $\overrightarrow{ON} = -\frac{4}{5}\mathbf{a}$.

4. (ii) $\cos(\angle BAC) = \frac{38}{50}$.
   Area $= 2\sqrt{66}$ units^2

   (iii) $\mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix} = 24$ or $\mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix} = 2$.

   (iv) $\frac{22}{3}$ units^3.

5. (i) $2x + 5y - 4z = -2$.

   (iii) $\mathbf{r} = \begin{pmatrix} 2 \\ 6 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 19 \\ -5 \\ -8 \end{pmatrix}, \lambda \in \mathbb{R}$.

6. $\lambda = 2.8, \mu = -1$.

7. $4\,|\mathbf{a}|^2$.

8. (i) $A(15, 80, 20)$.

   (iii) $B(10, 90, -45)$.

   (iv) 66.0 units.

   (v) $9.8°$.

9. (iii) It is the perpendicular distance from point P to the plane Π.