1 Cartesian arithmetic

Evaluate the following:

1. .... Powers of i ....

(a) i^{62} (b) i^{72} (c) i^{89} (d) i^{18} (e) i^{86} (f) i^{22} (g) i^{71} (h) i^{84}

2. .... Multiplication ....

(a) $7(2-3i)$
(b) $(-3+7i)(-7+3i)$
(c) $(-6+5i)(-5i)$
(d) $(7-5i)(9+3i)$
(e) $(9+3i)(-2+7i)$
(f) $(6-5i)(5+7i)$
(g) $(2-7i)(-3+5i)$
(h) $(-8+6i)(4)$
(i) $(7+9i)(-6)$

3. .... Division ....

(a) $\frac{-8+6i}{3-7i}$
(b) $\frac{5i}{-7-9i}$
(c) $\frac{-5+7i}{3-i}$
(d) $\frac{4-6i}{2+9i}$
(e) $\frac{-2+i}{1-2i}$
(f) $\frac{4i}{-3+6i}$
(g) $\frac{2i}{6}$
(h) $\frac{-2+7i}{8+8i}$
(i) $\frac{-4+8i}{2i}$

4. .... Powers ....

(a) $(-1+3i)^2$
(b) $(-5-2i)^3$
(c) $(1+8i)^2$
(d) $(-7-7i)^3$
(e) $(6-5i)^3$
(f) $(-2+i)^3$
(g) $(8-9i)^2$
(h) $(-9+3i)^2$
(i) $(1+3i)^2$

2 Linear/Quadratic equations

Solve the following equations:

5. .... Linear equations ....

(a) $7iz+6-2i=4$
(b) $(5+3i)z+9-7i=6+i$
(c) $(1-4i)z-2+4i=7$
(d) $(8+6i)z+2+9i=8+4i$
(e) $(8-3i)z+9+i=6+i$
(f) $(5+8i)z+8-9i=1+9i$
(g) $5iz+7i=7+7i$
(h) $(2+4i)z-7+2i=7+5i$

6. .... Quadratic equations ....

(a) $z^2 - 4z + 6 = 0$
(b) $9z^2 - 2z + 2 = 0$
(c) $8z^2 - z + 5 = 0$
(d) $6z^2 - 5z + 7 = 0$
(e) $9z^2 - 9z + 6 = 0$
(f) $3z^2 - 8z + 9 = 0$
(g) $7z^2 + 8z + 3 = 0$
(h) $7z^2 + 8z + 4 = 0$

3 Comparing real/imaginary parts

7. (a) The complex number z is such that $-16z + zz^* = -60 - 32i$, where z^* is the complex conjugate of z.
Find z in the form $x + yi$, where x and y are real. [4]

(b) The complex number z is such that $10iz + zz^* = 56 + 90i$, where z^* is the complex conjugate of z.
Find z in the form $x + yi$, where x and y are real. [4]

(c) The complex number z is such that $18z^* + zz^* = -56 - 90i$, where z^* is the complex conjugate of z.
Find z in the form $x + yi$, where x and y are real. [4]

(d) The complex number z is such that $-6z + zz^* = 55 + 48i$, where z^* is the complex conjugate of z.
Find z in the form $x + yi$, where x and y are real. [4]

4 Polynomial equations

8. (a) An equation is given by $2z^3 + 19z^2 + 54z + 55 = 0$.
   i. Verify that $z = -2 - i$ is a root of the equation.
   ii. Hence solve the equation.

(b) An equation is given by $2z^3 + 3z^2 - 70z + 150 = 0$.
   i. Verify that $z = 3 - i$ is a root of the equation.
   ii. Hence solve the equation.

9. (a) We have an equation $2z^3 + bz^2 + cz - 145 = 0$ and it is given that $z = -5 + 2i$ is a root of the equation.
Find, in either order, the values of the real constants b and c, and all the roots of the equation.

(b) We have an equation $z^3 + bz^2 + cz - 366 = 0$ and it is given that $z = 6 - 5i$ is a root of the equation.
Find, in either order, the values of the real constants b and c, and all the roots of the equation.

10. (a) We have an equation $z^4 + 28z^3 + 424z^2 + 3192z + 12740 = 0$. Given that $z = -7 - 9i$ is a root, solve the equation.
    (b) We have an equation $z^4 + 22z^3 + 195z^2 + 694z + 890 = 0$. Given that $z = -8 - 5i$ is a root, solve the equation.
    (c) We have an equation $z^4 - 18z^3 + 146z^2 - 242z + 113 = 0$. Given that $z = 8 + 7i$ is a root, solve the equation.
    (d) We have an equation $z^4 - 12z^3 + 20z^2 + 436z - 445 = 0$. Given that $z = 8 + 5i$ is a root, solve the equation.

11. (a) We have an equation $z^2 + (-9 + 3i)z + 48 - 59i = 0$. Given that $z = 8 + 5i$ is a root, solve the equation.
    (b) We have an equation $z^2 + (-13i)z - 89 + 21i = 0$. Given that $z = -7 + 8i$ is a root, solve the equation.
    (c) We have an equation $iz^2 + (-12 - 13i)z + 77 + 7i = 0$. Given that $z = 6 - 5i$ is a root, solve the equation.
    (d) We have an equation $iz^2 + (-1 - i)z - 4 + 20i = 0$. Given that $z = 1 - 5i$ is a root, solve the equation.

5 Complex numbers in polar form

12. .... Conversion to polar form ....

Convert the following to exponential form $re^{i\theta}$

Leave non-exact answers to 3 significant figures

(a) $-6 - i$ (b) $-6 + 3i$ (c) $3i$ (d) $6 - 3i$ (e) $7i$ (f) $7 + 7i$ (g) $-3 - i$ (h) $-6 - 8i$ (i) $4 + 4i$ (j) $8 + 7i$ (k) $8 + 7i$ (l) $3 - 4i$

13. .... Conversion to polar form (special values) ....

Convert the following to exponential form $re^{i\theta}$, where r and θ are exact values

(a) $-\frac{3}{2}\sqrt{3} - \frac{3}{2}i$ (b) $-\frac{5}{2} - \frac{5}{2}\sqrt{3}i$ (c) $2\sqrt{2} - 2\sqrt{2}i$ (d) $\frac{1}{2}\sqrt{3} + \frac{1}{2}i$ (e) $-\frac{3}{2} + \frac{3}{2}\sqrt{3}i$ (f) $\frac{5}{2} - \frac{5}{2}\sqrt{3}i$ (g) $\frac{9}{2} - \frac{9}{2}\sqrt{3}i$ (h) $2\sqrt{3} + 2i$ (i) -1 (j) $-\frac{5}{2}\sqrt{3} - \frac{5}{2}i$ (k) $-7i$ (l) $1 - \sqrt{3}i$

14. .... Conversion to cartesian form ....

Convert the following to cartesian form $x + yi$

Leave non-exact answers to 3 significant figures

(a) $5e^{-1.16i}$ (b) $5e^{2.43i}$ (c) $6e^{2.22i}$ (d) $3e^{-0.6i}$ (e) $8e^{1.72i}$ (f) $8e^{-2.84i}$ (g) $2e^{-0.31i}$ (h) $e^{-2.25i}$ (i) $e^{-1.12i}$ (j) $4e^{-2.32i}$ (k) $3e^{-1.93i}$ (l) $2e^{-2.63i}$

15. Conversion to cartesian form (special values)

Convert the following to cartesian form $x + yi$, where x and y are exact values

(a) $7e^{-\frac{\pi}{6}i}$
(b) $5e^{-\frac{\pi}{6}i}$
(c) $2e^{\frac{\pi}{2}i}$
(d) $7e^{-\frac{\pi}{6}i}$
(e) $8e^{\frac{5\pi}{6}i}$
(f) $3e^{\frac{5\pi}{6}i}$
(g) $3e^{-\frac{\pi}{2}i}$
(h) $4e^{-\frac{\pi}{2}i}$
(i) $6e^{\frac{5\pi}{6}i}$
(j) $8e^{-\frac{2\pi}{3}i}$
(k) $5e^{\frac{3\pi}{4}i}$
(l) $6e^{\frac{2\pi}{3}i}$

16. Polar form arithmetic

Evaluate the following

(a) $\dfrac{\left((2e^{\frac{11\pi}{12}i})^*e^{-\frac{\pi}{3}i}\right)^5}{e^{\frac{\pi}{2}i}}$

(b) $\dfrac{\left(e^{\frac{3\pi}{4}i}\left(e^{\frac{7\pi}{12}i}\right)^3\right)^*}{e^{-\frac{2\pi}{3}i}}$

(c) $\dfrac{2e^{\frac{5\pi}{12}i}\left(e^{\frac{\pi}{12}i}\right)^3}{\left(e^{-\frac{\pi}{12}i}\right)^*}$

(d) $\dfrac{\left((2e^{-\frac{2\pi}{3}i})^*e^{\frac{\pi}{6}i}\right)^8}{e^{\frac{\pi}{3}i}}$

(e) $\left(\dfrac{\left(2e^{\frac{5\pi}{6}i}\right)^9 e^{\frac{\pi}{12}i}}{e^{\pi i}}\right)^*$

(f) $\dfrac{e^{\frac{\pi}{4}i}\left(e^{-\frac{5\pi}{12}i}\right)^*}{\left(e^{-\frac{2\pi}{3}i}\right)^3}$

(g) $\dfrac{\left(e^{-\frac{\pi}{3}i}\left(e^{-\frac{\pi}{6}i}\right)^9\right)^*}{e^{\frac{\pi}{3}i}}$

(h) $\dfrac{\left(2e^{-\frac{\pi}{3}i}\left(e^{-\frac{\pi}{12}i}\right)^8\right)^*}{e^{\frac{\pi}{3}i}}$

(i) $\dfrac{2e^{\pi i}\left(e^{\frac{\pi}{6}i}\right)^2}{\left(e^{-\frac{2\pi}{3}i}\right)^*}$

17. Conditions for real/purely imaginary

Find the smallest positive integer value of n such that

(a) $\left(e^{-\frac{11\pi}{12}i}\right)^n$ is purely imaginary.
(b) $\left(e^{-\frac{\pi}{6}i}\right)^n$ is purely imaginary.
(c) $\left(e^{\frac{5\pi}{6}i}\right)^n$ is real and negative.
(d) $\left(e^{-\frac{5\pi}{12}i}\right)^n$ is purely imaginary.
(e) $\left(e^{-\frac{5\pi}{6}i}\right)^n$ is purely imaginary.
(f) $\left(e^{\frac{\pi}{12}i}\right)^n$ is purely imaginary.
(g) $\left(e^{-\frac{3\pi}{4}i}\right)^n$ is purely imaginary.
(h) $\left(e^{\frac{\pi}{6}i}\right)^n$ is purely imaginary.

6 Mixed practice and the Argand diagram

18. The complex number z is given by $z = a + 7i$, where a is a real number.

(a) Find the possible values of a if $\dfrac{z^2}{z^*}$ is purely imaginary

For the rest of the equation it is further given that $a > 0$.

(b) Find the smallest integer value of n such that $|z^n| \geq 1000$.
(c) For the value of n found in part (b), find the values of $|z^n|$ and $\arg(z^n)$, where $-\theta < \arg(z^n) \leq \theta$.
(d) On a single Argand diagram mark out the points A, B, C, D, E representing the complex number $z, \dfrac{z^2}{z^*}, z^*, \dfrac{98}{z}, \dfrac{z^2}{14}$ respectively.

19. The complex number z is given by $z = a - 3i$, where a is a real number.
    (a) Find the possible values of a if z^3 is purely imaginary

    For the rest of the equation it is further given that $a > 0$.

    (b) Find the smallest integer value of n such that $|z^n| \geq 1000$.
    (c) For the value of n found in part (b), find the values of $|z^n|$ and $\arg(z^n)$, where $-\theta < \arg(z^n) \leq \theta$.
    (d) On a single Argand diagram mark out the points A, B, C, D, E representing the complex number $z, \frac{z^3}{36}, z^*, \frac{18}{z}, \frac{z^2}{6}$ respectively.

20. The complex number z is given by $z = 9 + bi$, where b is a real number.
    (a) Find the possible values of b if $\frac{z^2}{z^*}$ is real.

    For the rest of the equation it is further given that $b > 0$.

    (b) Find the smallest integer value of n such that $|z^n| \geq 1000$.
    (c) For the value of n found in part (b), find the values of $|z^n|$ and $\arg(z^n)$, where $-\theta < \arg(z^n) \leq \theta$.
    (d) On a single Argand diagram mark out the points A, B, C, D, E representing the complex number $z, \frac{z^2}{z^*}, z^*, \frac{162}{z}, \frac{z^2}{18}$ respectively.

21. The complex number z is given by $z = -9 + bi$, where b is a real number.
    (a) Find the possible values of b if z^3 is real.

    For the rest of the equation it is further given that $b > 0$.

    (b) Find the smallest integer value of n such that $|z^n| \geq 1000$.
    (c) For the value of n found in part (b), find the values of $|z^n|$ and $\arg(z^n)$, where $-\theta < \arg(z^n) \leq \theta$.
    (d) On a single Argand diagram mark out the points A, B, C, D representing the complex number $z, \frac{z^3}{324}, z^*, \frac{162}{z}$ respectively.

Answers

1. (a) -1 (b) 1 (c) i (d) -1 (e) -1 (f) -1 (g) $-i$ (h) 1

2. (a) $14-21i$ (b) $-58i$ (c) $25+30i$ (d) $78-24i$ (e) $-39+57i$ (f) $65+17i$ (g) $29+31i$ (h) $-32+24i$ (i) $-42-54i$

3. (a) $\frac{-33-19i}{29}$ (b) $\frac{-9-7i}{26}$ (c) $\frac{-11+8i}{5}$ (d) $\frac{-46-48i}{85}$ (e) $\frac{-4-3i}{5}$ (f) $\frac{8-4i}{15}$ (g) $\frac{i}{3}$ (h) $\frac{5+9i}{16}$ (i) $4+2i$

4. (a) $-8-6i$ (b) $-65-142i$ (c) $-63+16i$ (d) $686-686i$ (e) $-234-415i$ (f) $-2+11i$ (g) $-17-144i$ (h) $72-54i$ (i) $-8+6i$

5. (a) $z=\frac{2+2i}{7}$ (b) $z=\frac{9+49i}{34}$ (c) $z=\frac{25+32i}{17}$ (d) $z=\frac{9-38i}{50}$ (e) $z=\frac{-24-9i}{73}$ (f) $z=\frac{109+146i}{89}$ (g) $z=\frac{-7i}{5}$ (h) $z=\frac{4-5i}{2}$

6. (a) $z=2\pm\sqrt{2}i$ (b) $z=\frac{1}{9}\pm\frac{1}{9}\sqrt{17}i$ (c) $z=\frac{1}{16}\pm\frac{1}{16}\sqrt{159}i$ (d) $z=\frac{5}{12}\pm\frac{1}{12}\sqrt{143}i$ (e) $z=\frac{1}{2}\pm\frac{1}{6}\sqrt{15}i$ (f) $z=\frac{4}{3}\pm\frac{1}{3}\sqrt{11}i$ (g) $z=-\frac{4}{7}\pm\frac{1}{7}\sqrt{5}i$ (h) $z=-\frac{4}{7}\pm\frac{2}{7}\sqrt{3}i$

7. (a) $z=8+2i$ (b) $z=9+5i$ (c) $z=-9+5i$ (d) $z=3-8i$

8. (a) $z=-2-i, z=-2+i$ or $z=-\frac{11}{2}$ (b) $z=3+i, z=3-i$ or $z=-\frac{15}{2}$

9. (a) $b=10, c=29$, $z=-5-2i, z=-5+2i$ or $z=\frac{5}{2}$ (b) $b=-12, c=61$, $z=6+5i, z=6-5i$ or $z=6$

10. (a) $z=-7+9i, z=-7-9i, z=-7+7i$ or $z=-7-7i$
    (b) $z=-8-5i, z=-8+5i, z=-3+i$ or $z=-3-i$
    (c) $z=8-7i, z=8+7i, z=1$ or $z=1$
    (d) $z=8-5i, z=8+5i, z=1$ or $z=-5$

11. (a) $z=8+5i$ or $z=1-8i$ (b) $z=-7+8i$ or $z=7+5i$ (c) $z=6-5i$ or $z=7-7i$ (d) $z=1-5i$ or $z=4i$

12. (a) $6.08e^{-2.98i}$ (b) $6.71e^{2.68i}$ (c) $3.00e^{1.57i}$ (d) $6.71e^{-0.464i}$ (e) $7.00e^{1.57i}$ (f) $9.90e^{0.785i}$ (g) $3.16e^{-2.82i}$ (h) $10.0e^{-2.21i}$ (i) $5.66e^{0.785i}$ (j) $10.6e^{0.719i}$ (k) $10.6e^{0.719i}$ (l) $5.00e^{-0.927i}$

13. (a) $3e^{-\frac{5\pi}{6}i}$ (b) $5e^{-\frac{2\pi}{3}i}$ (c) $4e^{-\frac{\pi}{4}i}$ (d) $e^{\frac{\pi}{6}i}$ (e) $3e^{\frac{2\pi}{3}i}$ (f) $5e^{-\frac{\pi}{3}i}$ (g) $9e^{-\frac{\pi}{3}i}$ (h) $4e^{\frac{\pi}{6}i}$ (i) $e^{\pi i}$ (j) $5e^{-\frac{5\pi}{6}i}$ (k) $7e^{-\frac{\pi}{2}i}$ (l) $2e^{-\frac{\pi}{3}i}$

14. (a) $2-4.58i$ (b) $-3.79+3.27i$ (c) $-3.63+4.78i$ (d) $2.48-1.69i$ (e) $-1.19+7.91i$ (f) $-7.64-2.38i$ (g) $1.9-0.61i$ (h) $-0.628-0.778i$ (i) $0.436-0.9i$ (j) $-2.72-2.93i$ (k) $-1.05-2.81i$ (l) $-1.74-0.979i$

15. (a) $\frac{7}{2}\sqrt{3}-\frac{7}{2}i$ (b) $\frac{5}{2}\sqrt{3}-\frac{5}{2}i$ (c) $2i$ (d) $\frac{7}{2}\sqrt{3}-\frac{7}{2}i$ (e) $-4\sqrt{3}+4i$ (f) $-\frac{3}{2}\sqrt{3}+\frac{3}{2}i$ (g) $-3i$ (h) $-4i$ (i) $-3\sqrt{3}+3i$ (j) $-4-4\sqrt{3}i$ (k) $-\frac{5}{2}\sqrt{2}+\frac{5}{2}\sqrt{2}i$ (l) $-3+3\sqrt{3}i$

16. (a) $32e^{\frac{5\pi}{12}i}$ (b) $e^{-\frac{5\pi}{6}i}$ (c) $2e^{\frac{3\pi}{4}i}$ (d) $256e^{-\frac{\pi}{3}i}$ (e) $512e^{\frac{7\pi}{12}i}$ (f) $e^{-\frac{\pi}{6}i}$ (g) $e^{-\frac{\pi}{6}i}$ (h) $2e^{\frac{2\pi}{3}i}$ (i) $2e^{0i}$

17. (a) 6 (b) 3 (c) 6 (d) 6 (e) 3 (f) 6 (g) 2 (h) 3

18. (a) $a=0, 7\sqrt{3},$ or $-7\sqrt{3}$
(b) Smallest $n=3$
(c) $|z^n|=2744,\ \arg(z^n)=\frac{1}{2}\pi$

19. (a) $a=0, 3\sqrt{3},$ or $-3\sqrt{3}$
(b) Smallest $n=4$
(c) $|z^n|=1296,\ \arg(z^n)=-\frac{2}{3}\pi$

20. (a) $b=0, 9\sqrt{3},$ or $-9\sqrt{3}$
(b) Smallest $n=3$
(c) $|z^n|=5832,\ \arg(z^n)=\pi$

21. (a) $b=0, 9\sqrt{3},$ or $-9\sqrt{3}$
(b) Smallest $n=3$
(c) $|z^n|=5832,\ \arg(z^n)=0$

For the Argand diagrams (part d) refer to math-atlas.vercel.app/questions/qn1305 with the UQNs 170, 231, 390, 491 for questions 18-21 respectively