2017 Specimen Paper 1

1. A circular ink-blot is expanding such that the rate of change of the diameter D with respect to time t is 0.25 cm/s. Find the rate of change of both the circumference and the area of the circle with respect to t when the radius of the circle is 1.5 cm. Give your answer correct to 4 decimal places. [4]

2. The curve C with equation $y = x^3$ is transformed onto the curve with equation $y = f(x)$ by a translation of 2 units in the negative x-axis direction, followed by a stretch of factor $\frac{1}{2}$ parallel to the y-axis, followed by a translation of 1 unit in the positive y-direction.

   (a) Write down the equation of the new curve. [1]

   (b) Sketch C and the curve with equation $y = f(x)$ on the same diagram, stating the exact value of the coordinates of the points where $y = f(x)$ crosses the x- and y-axes. Find the x-coordinate of the point(s) where the two curves intersect, giving your answer(s) correct to 3 decimal places. [4]

3. (a) Sketch the curve with equation $y = \dfrac{x^2 - 12}{x}$, giving the exact coordinates of the point(s) where the curve crosses the axes and the equations of any asymptotes. [4]

   (b) Hence, or otherwise, solve the inequality $\dfrac{x^2 - 12}{x} < 1$. [3]

4. A science student is investigating the elasticity of a new compound She drops a ball made of the new compound vertically onto a hard surface and measures the height reached by the ball after each successive bounce. She drops the ball from an initial height of 200 cm and she estimates that the height the ball reaches after each bounce is $\frac{8}{9}$ of the height reached by the previous bounce.

   (a) Find the total distance that the ball has travelled when it reaches the highest point after the fourth bounce. Give your answer correct to the nearest centimetre. [2]

   (b) The ball is considered to have stopped bouncing when a bounce first results in the height the ball reaches being less than 0.01 cm. Find how many bounces the ball has made and the total distance that the ball has travelled in this case. Give your answer correct to the nearest centimetre. [6]

5. The curve C has equation $y = \dfrac{1}{x}(\ln x)^3$, where $x > 1$.

   (a) Find the exact x-coordinate, $x = x_1$, of the turning point on C and explain whether it is a maximum or a minimum turning point. [4]

   (b) Without using a calculator, find the exact area of the region between C, the x-axis and the lines with equations $x = e$ and $x = x_1$. [3]

6. (a) The non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$. Given that $\mathbf{b} \neq -\mathbf{c}$, find a linear relationship between $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. [3]

(b) The variable vector $\mathbf{v}$ satisfies the equation $\mathbf{v} \times (\mathbf{i} - 3\mathbf{k}) = 2\mathbf{j}$. Find the set of vectors $\mathbf{v}$ and describe this set geometrically. [5]

7. **Do not use a calculator in answering this question.**

(a) Showing your working, find the complex numbers z and w which satisfy the simultaneous equations

$$2iz + (1 - 2i)w = 4 \text{ and}$$
$$(1 + i)z + (2 + i)w = 3.$$

[6]

(b) The complex number u is given by $u = \cos\theta + i\sin\theta$, where $0 < \theta < \pi$. Show that $1 - u^2 = -2iu\sin\theta$ and hence or otherwise find the modulus and argument of $1 - u^2$ in terms of θ. [5]

8. The asteroid, a curve C which is used to characterise various properties of energy and magnetism, has parametric equation

$$x = a\cos^3 t, y = a\sin^3 t,$$

where $0 \leq t \leq \frac{1}{2}\pi$ and a is a positive constant.

(a) Find the equation of the tangent to C at the point P with parameter p. [3]

(b) The tangent at P meets the x-axis at the point A and meets the y-axis at the point B. Show that the length AB depends only on a. [3]

It is given that $a = 1$.

(c) Find a Cartesian equation of C. [2]

(d) The region bounded by C and the x- and y-axes is rotated through $360°$ about the y-axis. Find the exact value of the volume of revolution of the solid formed. [4]

9. A man M is at the top of a mountain which is of height h km. The radius of the earth is assumed to be a constant R km. The furthest point on the earth's surface that the man can see is a point P such that $MP = x$ km and the angle $POM = \theta$ where O is the centre of the earth (see diagram). You may assume that the height of the man is negligible.
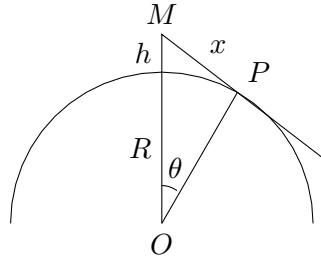
(a) Show that $x = (2hR)^{\frac{1}{2}}\left(1 + \dfrac{h}{2R}\right)^{\frac{1}{2}}$. [3]

(b) It is given that h is small compared to R. Show that, if $\alpha = \dfrac{h}{R}$,

$$\sin\theta \approx (2\alpha)^{\frac{1}{2}}\left(1 - \frac{3}{4}\alpha\right).$$

[5]

(c) The man M has a scientific instrument which enables him to estimate the angle between PM and the horizontal. Given that this angle is $2°$ and that the radius of the earth is 6357 km, find estimates for the values α and h. [4]

10. The point A has coordinates $(-1, 2, -1)$. The line l has equation $\dfrac{x}{2} = \dfrac{y+1}{-3} = \dfrac{z-2}{1}$.

(a) Find the Cartesian equation of the plane π which contains A and is perpendicular to l. [3]

(b) Hence, or otherwise, find the coordinates of point P on l which is closest to A. [3]

(c) The line m passes through the point with coordinates $(4, -5, 10)$ and P. The line n lies in the same plane as l and m. Find a Cartesian equation for n if n is the reflection of the line m about the line l. [6]

3

11. A pond has a surface area of 10 m^2. Biologists have planted an area of new weeds. They estimate how many weeds there are and the rate at which they are spreading by finding the area of the pond the weeds cover at various times. They believe that the area, A m^2, of weeds present at time t months is such that the rate at which the area is increasing is proportional to the product of the area of pond covered by the weeds and the area of the pond not covered by the weeds. It is known that the initial area of weeds is 2 m^2 and that the area of weeds is 4 m^2 after 5 months.

(a) Write down a differential equation expressing the relation between A and t. Find the time at which 80% of the pond is covered in weeds, giving your answer correct to 2 decimal places. [8]

(b) Given that the experiment is stopped after 2 years, find the area of pond covered by weeds, giving your answer correct to 2 decimal places. [2]

(c) Write the solution of the differential equation in the form $A = f(t)$ and sketch the curve. [4]

Numerical answers

1. Rate of change of circumference $= 0.7854$ cm/s.
   Rate of change of area $= 0.5890$ cm^2/s.

2. (a) $y = \frac{1}{2}(x+2)^2 + 1$.
   (b) $x = 7.722$.

3. (b) $x < -3$ or $0 < x < 4$.

4. (a) 1277 cm.
   (b) 3400 cm.

5. (a) $x_1 = e^3$, minimum point.
   (b) 20 units^2.

6. (a) $\mathbf{a} = \lambda(\mathbf{b} + \mathbf{c}), \lambda \in \mathbb{R}$.
   (b)
   $$\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}, \qquad \mu \in \mathbb{R}.$$

   The set of vectors $\mathbf{v}$ form a line passing through the point $(2, 0, 0)$ with direction vector $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$.

7. (a) $w = \frac{3+i}{5}, z = \frac{1-3i}{2}$.
   (b) $|1 - u^2| = 2\sin\theta$.
   $\arg(1 - u^2) = \theta - \frac{\pi}{2}$.

8. (a) $y = -x\tan p + a\sin p$.
   (b) Length of $AB = a$ units which depends only on a.
   (c) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.
   (d) $\frac{16\pi}{105}$ units^3.

9. (c) $\alpha \approx 6.10 \times 10^{-4}, h \approx 3.87$ km.

10. (a) $2x - 3y + z = 9$.
    (b) $P(-2, 2, 1)$.
    (c) $\mathbf{r} = (-2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \phi(6\mathbf{i} - 11\mathbf{j} - 3\mathbf{k})$.

11. (a) 14.13 months.
    (b) 9.65 m^2.
    (c) $A = \dfrac{10\left(\frac{8}{3}\right)^{\frac{t}{5}}}{4 + \left(\frac{8}{3}\right)^{\frac{t}{5}}}$.

Worked Solutions

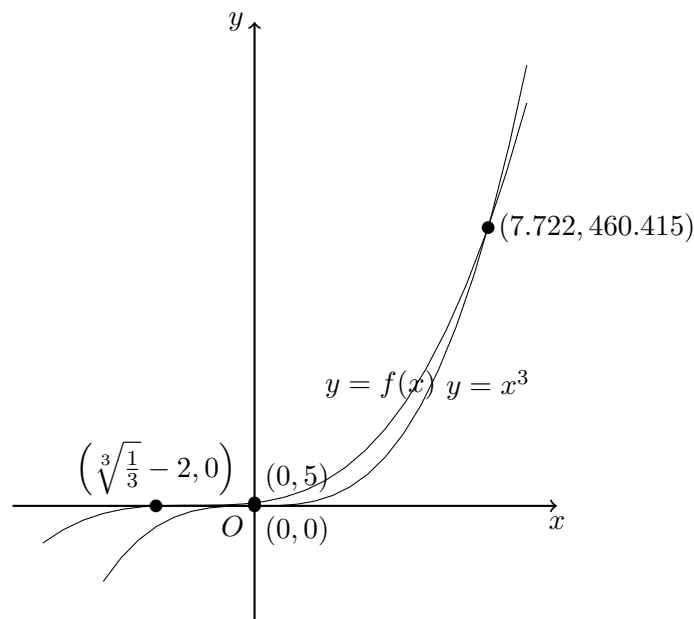1. Let r, C and A represent the radius, circumference and the area, respectively, of the ink-blot.

$$\frac{\mathrm{d}D}{\mathrm{d}t} = 0.25$$
$$C = \pi D.$$
$$\frac{\mathrm{d}C}{\mathrm{d}D} = \pi$$
$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\mathrm{d}C}{\mathrm{d}D} \times \frac{\mathrm{d}D}{\mathrm{d}t}$$
$$= \frac{\pi}{4}$$
$$= 0.7854 \text{ cm/s (4dp)}$$

$$A = \pi r^2$$
$$= \frac{\pi D^2}{4}$$
$$\frac{\mathrm{d}A}{\mathrm{d}D} = \frac{\pi D}{2}$$
$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}D} \times \frac{\mathrm{d}D}{\mathrm{d}t}$$
$$= \frac{\pi D}{8}$$
$$= \frac{1.5\pi}{8}$$
$$= 0.5890 \text{ cm}^2/\text{s (4dp)}$$
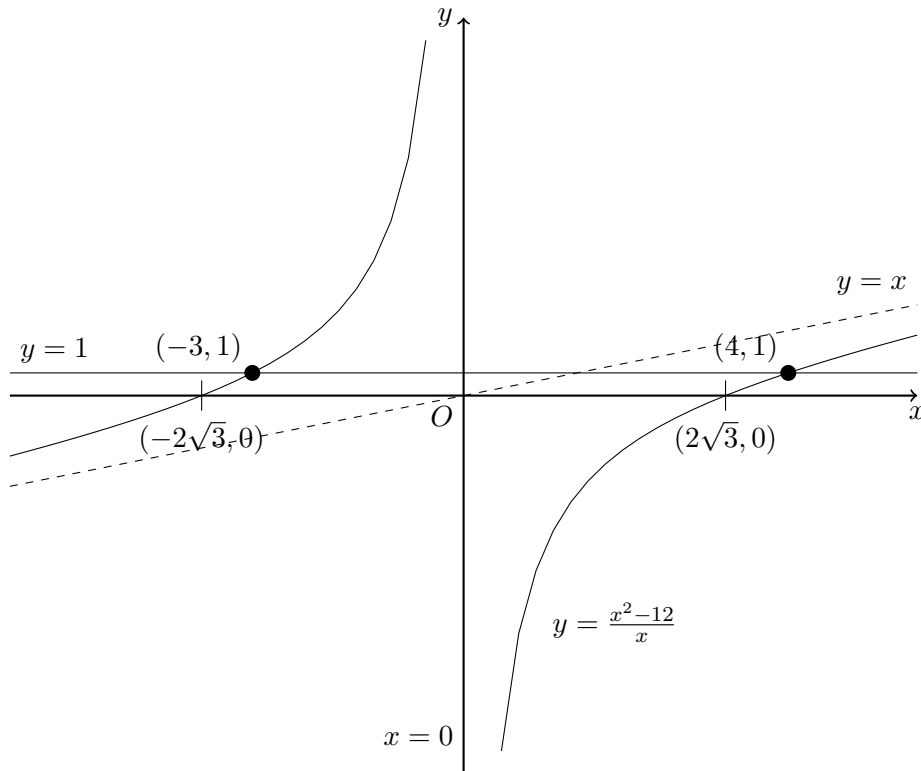
Rate of change of circumference $= 0.7854$ cm/s.
Rate of change of area $= 0.5890$ cm^2/s.

2. (a) $y = x^3 \to y = (x+2)^3 \to y = \frac{1}{2}(x+2)^3 \to y = \frac{1}{2}(x+2)^3 + 1$.
   Equation of new curve: $y = \frac{1}{2}(x+2)^3 + 1$.
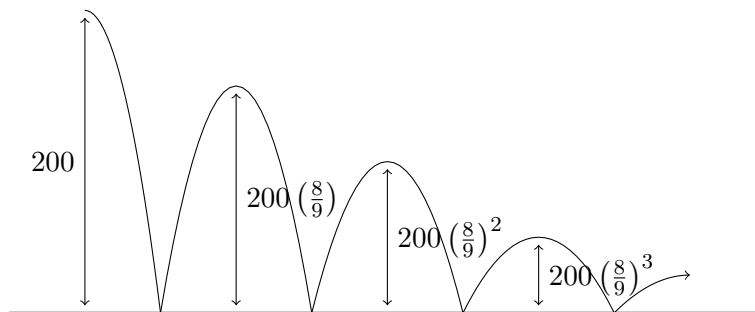
   (b) x-coordinate of point of intersection $= 7.722$.

3. (a) Asymptotes: $y = x$ and $x = 0$.
   x-intercepts: $(-2\sqrt{3}, 0)$ and $(2\sqrt{3}, 0)$.

$y = 1$ $(-3, 1)$ $(4, 1)$ $y = x$

$(-2\sqrt{3}, 0)$ $(2\sqrt{3}, 0)$

$y = \frac{x^2 - 12}{x}$

$x = 0$

(b) Using the graph, $x < -3$ or $0 < x < 4$.

4. (a) Up until the highest point after the fourth bounce,

200

$200 \left(\frac{8}{9}\right)$

$200 \left(\frac{8}{9}\right)^2$

$200 \left(\frac{8}{9}\right)^3$

$$\text{Distance travelled} = 200 + 2(200)\left(\frac{8}{9}\right) + 2(200)\left(\frac{8}{9}\right)^2 + 2(200)\left(\frac{8}{9}\right)^3 + (200)\left(\frac{8}{9}\right)^4$$

$$= 1277 \text{ cm (nearest cm)}$$

(b) Let u_n represent the height of the ball **before** the n-th bounce. u_n forms a geometric progression with first term 200 and common difference $\frac{8}{9}$.

$$u_n < 0.01$$

$$200\left(\frac{8}{9}\right)^{n-1} < 0.01$$

$$\left(\frac{8}{9}\right)^{n-1} < 0.00005$$

7

$$(n-1)\ln\frac{8}{9} < \ln 0.00005$$

$$n - 1 > 84.08 \qquad (\text{since } \ln\frac{8}{9} < 0)$$

$$n > 85.08$$

Hence the ball is considered to have stopped bouncing on the 86-th bounce. We calculate the total distance travelled using two geometric progressions. The first is for the distance travelled downwards with first term 200, common ratio $\frac{8}{9}$ and 86 terms. The second is for the distance travelled upwards with first term $200\left(\frac{8}{9}\right)^2$, common ratio $\frac{8}{9}$ and 85 terms.

$$\text{Total distance} = \frac{200\left(1 - \left(\frac{8}{9}\right)^{86}\right)}{1 - \frac{8}{9}} + \frac{200\left(\frac{8}{9}\right)\left(1 - \left(\frac{8}{9}\right)^{85}\right)}{1 - \frac{8}{9}}$$

$$= 3400 \text{ cm}$$

5. (a) $y = \dfrac{(\ln x)^3}{x}$

$\dfrac{dy}{dx} = \dfrac{x \cdot 3(\ln x)^2 \frac{1}{x} - (\ln x)^3}{x^2}$

$\qquad = \dfrac{(\ln x)^2(3 - \ln x)}{x^2}$

At turning point, $\frac{dy}{dx} = 0$.

$\dfrac{(\ln x)^2(3 - \ln x)}{x^2} = 0$

$(\ln x)^2(3 - \ln x) = 0$

$\qquad \ln x = 0 \qquad \text{or} \qquad \ln x = 3$

$x = 1 \text{ (N.A. as } x > 1) \qquad x = e^3$

Hence $x_1 = e^3$.

When $1 < x < e^3$, $\frac{dy}{dx} < 0$.

When $x > e^3$, $\frac{dy}{dx} > 0$.

Hence it is a minimum point when $x = e^3$.

(b) Area $= \displaystyle\int_{e}^{e^3} \frac{1}{x}(\ln x)^3 \, dx$

$\qquad = \left[\dfrac{(\ln x)^4}{4}\right]_{e}^{e^3}$

$\qquad = \dfrac{(\ln e^3)^4 - (\ln e)^4}{4}$

$\qquad = \dfrac{3^4 - 1}{4}$

$\qquad = 20 \text{ units}^2.$

6. (a) $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$

$\mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{a} = \mathbf{0}$

$\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{0}$

$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{0}$

Since $\mathbf{a}, \mathbf{b} + \mathbf{c} \neq \mathbf{0}$, $\mathbf{a}$ is parallel to $\mathbf{b} + \mathbf{c}$.

Hence $\mathbf{a} = \lambda(\mathbf{b} + \mathbf{c})$ where $\lambda \in \mathbb{R}$.

(b) Let $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3y \\ 3x+z \\ -y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$y = 0 \tag{1}$$
$$3x + z = 0 \tag{2}$$
$$y = 0 \tag{3}$$

Solving using a GC,

$$\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}, \qquad \mu \in \mathbb{R}.$$

Hence the set of vectors $\mathbf{v}$ forms a line passing through the point $(2, 0, 0)$ with direction vector $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$.

7. (a) $2iz + (1 - 2i)w = 4$

$z = \dfrac{4 - (1 - 2i)w}{2i}$

Substituting into equation (2),

$(1 + i)\dfrac{4 - (1 - 2i)w}{2i} + (2 + i)w = 3.$

$\dfrac{1 + i}{2i} \cdot \dfrac{i}{i}(4 - (1 - 2i)w) + (2 + i)w = 3.$

$\dfrac{-1 + i}{-2}(4 - (1 - 2i)w) + (2 + i)w = 3.$

$-2(-1 + i) + \dfrac{-1 + i}{2}(1 - 2i)w + (2 + i)w = 3.$

$\dfrac{-1 + i}{2}(1 - 2i)w + (2 + i)w = 3 - 2 + 2i.$

$\dfrac{1 + 3i}{2}w + (2 + i)w = 1 + 2i.$

$(1 + 3i + 2(2 + i))w = 2(1 + 2i).$

$(5 + 5i)w = 2 + 4i.$

$w = \dfrac{2 + 4i}{5(1 + i)} \cdot \dfrac{1 - i}{1 - i}.$

$= \dfrac{6 - 2i}{5(2)}$

$= \dfrac{3 + i}{5}$

9

Substituting $w = \dfrac{3+i}{5}$,

$$z = \frac{4 - (1-2i)\left(\frac{3+i}{5}\right)}{2i}.$$

$$= \frac{20 - (5-5i)}{10i} \cdot \frac{i}{i}$$

$$= \frac{-1+3i}{-2}$$

$$= \frac{1-3i}{2}$$

Hence $w = \dfrac{3+i}{5},\ z = \dfrac{1-3i}{2}.$

(b)
$$1 - u^2 = 1 - \left(e^{i\theta}\right)^2 \qquad .$$

$$= 1 - e^{i2\theta}$$

$$= e^{i\theta}e^{-i\theta} - e^{i\theta}e^{i\theta}$$

$$= e^{i\theta}\left(e^{-i\theta} - e^{i\theta}\right)$$

$$= u(\cos\theta - i\sin\theta - (\cos\theta + i\sin\theta))$$

$$= u(-2i\sin\theta)$$

$$= -2iu\sin\theta$$

(c)
$$|1 - u^2| = |-2i| \cdot |u| \cdot |\sin\theta|$$

$$= 2(1)\sin\theta$$

$$= 2\sin\theta$$

since $|u| = 1$ and $0 < \theta < \pi \Rightarrow \sin\theta > 0$.

$$\arg(1 - u^2) = \arg(-2i) + \arg(u) + \arg(\sin\theta).$$

$$= -\frac{\pi}{2} + \theta + 0$$

$$= \theta - \frac{\pi}{2}$$

8. (a) $\dfrac{dy}{dt} = 3a\sin^2 t(\cos t)\qquad.$

$$\frac{dx}{dt} = 3a\cos^2 t(-\sin t)$$

$$\frac{dy}{dx} = \frac{dy}{dy}t \div \frac{dx}{dt}$$

$$= \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t}$$

$$= -\frac{\sin t}{\cos t}$$

$$= -\tan t.$$

At point $P(a\cos^3 p, a\sin^3 p)$, $\dfrac{dy}{dx} = -\tan p.$

$y - a\sin^3 p = -\tan p(x - a\cos^3 p).$

$y = -(\tan p)x + a\sin p\cos^2 p + a\sin^3 p$

$y = -(\tan p)x + a\sin p(\cos^2 p + \sin^2 p).$

Equation of tangent at P: $y = -(\tan p)x + a\sin p.$

(b) At A, $y = 0$.

$$x = \frac{-a\sin p}{-\tan p} = a\cos p.$$

At B, $x = 0$.

$y = a \sin p$.

$A(a \cos p, 0)$, $B(0, a \sin p)$.

Length of $AB = \sqrt{(a \cos p)^2 + (a \ sinp)^2}$

$\qquad\qquad = \sqrt{a^2(\sin^2 p + \cos^2 p)}$

$\qquad\qquad = a \qquad\qquad$ which depends only on a

(c) $a = 1$.

$\cos \theta = x^{\frac{1}{3}}$.

$\sin \theta = y^{\frac{1}{3}}$.

Since $\cos^2 \theta + \cos^2 \theta = 1$,

Cartesian equation of C: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

(d) When $x = 0, y = 1$. When $y = 0, x = 1$.

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1 \quad \Rightarrow \quad x^2 = \left(1 - y^{\frac{2}{3}}\right)^3$.

$\text{Volume of solid of revolution} = \pi \int_0^1 x^2 \, dy$

$\qquad\qquad = \pi \int_0^1 \left(1 - y^{\frac{2}{3}}\right)^3 \, dy$

$\qquad\qquad = \pi \int_0^1 1 - 3y^{\frac{2}{3}} + 3\left(y^{\frac{2}{3}}\right)^2 - \left(y^{\frac{2}{3}}\right)^3 \, dy$

$\qquad\qquad = \pi \int_0^1 1 - 3y^{\frac{2}{3}} + 3y^{\frac{4}{3}} - y^2 \, dy$

$\qquad\qquad = \pi \left[y - \frac{3}{\frac{5}{3}}y^{\frac{5}{3}} + \frac{3}{\frac{7}{3}}y^{\frac{7}{3}} - \frac{1}{3}y^3 \right]_0^1$

$\qquad\qquad = \frac{16\pi}{105} \text{ units}^3.$

9. (a) $x = \sqrt{(R+h)^2 - R^2}$

$\qquad = \sqrt{R^2 + 2hR + h^2 - R^2}$

$\qquad = \sqrt{2hR + h^2}$

$\qquad = (2hR)^{\frac{1}{2}} \left(1 + \frac{h}{2R}\right)^{\frac{1}{2}}.$

(b) $\sin\theta = \dfrac{x}{h+R}$

$$= \frac{(2hR)^{\frac{1}{2}}\left(1+\frac{h}{2R}\right)^{\frac{1}{2}}}{h+R}$$

$$= \frac{(2hR)^{\frac{1}{2}}}{h+R}\left(1+\frac{1}{2}\alpha\right)^{\frac{1}{2}}$$

$$= \frac{\frac{(2hR)^{\frac{1}{2}}}{R}}{\frac{h}{R}+1}\left(1+\frac{1}{4}\alpha+\cdots\right)$$

$$= (2\alpha)^{\frac{1}{2}}(1+\alpha)^{-1}\left(1+\frac{1}{4}\alpha+\cdots\right)$$

$$= (2\alpha)^{\frac{1}{2}}(1-\alpha+\cdots)\left(1+\frac{1}{4}\alpha+\cdots\right)$$

$$= (2\alpha)^{\frac{1}{2}}\left(1-\alpha+\frac{1}{4}\alpha+\cdots\right)$$

$$\approx (2\alpha)^{\frac{1}{2}}\left(1-\frac{3}{4}\alpha\right).$$

(c) $\sin 2° \approx (2\alpha)^{\frac{1}{2}}\left(1-\dfrac{3}{4}\alpha\right).$

Using GC, $\alpha \approx 6.0954 \times 10^{-4}$.

$\dfrac{h}{R} = \alpha \approx 6.0954 \times 10^{-4}.$

$h = 6357\alpha \approx 3.8748$ km.

$\alpha \approx 6.10 \times 10^{-4}, h \approx 3.87$ km.

10. (a) $l : \mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, where $\lambda \in \mathbb{R}$.

Normal vector of π: $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

$\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = -9.$

Hence equation of π: $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = -9.$

$2x - 3y + z = -9.$

(b) The intersection between π and l is the point P.

Substituting equation of l into equation of π:

$\begin{pmatrix} 2\lambda \\ -1-3\lambda \\ 2+\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = -9.$

$4\lambda + 3 + 9\lambda + 2 + \lambda = -9.$

$14\lambda = -14.$

$\lambda = -1.$

Hence, coordinates of $P = (-2, 2, 1).$

(c) Direction vector of m, $\overrightarrow{PC} = \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -7 \\ 9 \end{pmatrix}.$

Equation of $m : \mathbf{r} = \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ -7 \\ 9 \end{pmatrix}, \mu \in \mathbb{R}.$

Let N denote the foot of perpendicular from the point $C(4, -5, 10)$ to the line l and $\mathbf{d}$ the direction vector of l.

$$\overrightarrow{PN} = \frac{\overrightarrow{PC} \cdot \mathbf{d}}{|\mathbf{d}|^2} (\mathbf{d})$$

$$= \frac{\begin{pmatrix} 6 \\ -7 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}}{2^2 + 3^2 + 1^2} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$= \frac{12 + 21 + 9}{14} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -9 \\ 3 \end{pmatrix}$$

$$\overrightarrow{ON} - \overrightarrow{OP} = \begin{pmatrix} 6 \\ -9 \\ 3 \end{pmatrix}$$

$$\overrightarrow{ON} = \begin{pmatrix} 4 \\ -7 \\ 4 \end{pmatrix}.$$

Let C' denote the point obtained when C is reflected in line l.

By ratio theorem, $\overrightarrow{ON} = \dfrac{\overrightarrow{OC} + \overrightarrow{OC'}}{2}.$

$$\overrightarrow{OC'} = 2\overrightarrow{ON} - \overrightarrow{OC}$$

$$= \begin{pmatrix} 4 \\ -9 \\ -2 \end{pmatrix}$$

$$\overrightarrow{PC'} = \begin{pmatrix} 6 \\ -11 \\ -3 \end{pmatrix}.$$

Equation of $l : \mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} + \phi \begin{pmatrix} 6 \\ -11 \\ -3 \end{pmatrix}, \phi \in \mathbb{R}.$

11. (a) $\dfrac{\mathrm{d}A}{\mathrm{d}t} = kA(10 - A)$ where k is a constant.

$$\frac{1}{A(10-A)}\frac{\mathrm{d}A}{\mathrm{d}t} = k$$

$$\int \frac{1}{A(10-A)}\,\mathrm{d}A = \int k\,\mathrm{d}t$$

$$\frac{1}{10}\int \frac{1}{A} + \frac{1}{10-A}\,\mathrm{d}A = kt + C$$

$$\frac{1}{10}\left(\ln|A| - \ln|10-A|\right) = kt + C$$

$$\frac{1}{10}\ln\left|\frac{A}{10-A}\right| = kt + C$$

$$\ln\left|\frac{A}{10-A}\right| = 10kt + C'$$

$$\left|\frac{A}{10-A}\right| = \mathrm{e}^{10kt+C'}$$

$$\frac{A}{10-A} = B\mathrm{e}^{10kt} \quad \text{where } B = \pm\mathrm{e}^{C'}.$$

When $t = 0, A = 2 \quad \Rightarrow \quad B = \frac{1}{4}$.

When $t = 5, A = 4 \quad \Rightarrow \quad k = \frac{1}{50}\ln\frac{8}{3}$.

$$\frac{A}{10-A} = \frac{1}{4}\mathrm{e}^{\frac{t}{5}\ln\frac{8}{3}}.$$

When $A = 8, t = 24$. Hence $t = 5\dfrac{\ln 16}{\ln\frac{8}{3}} = 14.13$ months (2dp).

(b) When $t = 24$, $\frac{A}{10-A} = \frac{1}{4}\mathrm{e}^{\frac{24}{5}\ln\frac{8}{3}}$.

$A = 9.65$ m^2 (2dp).

(c) $\dfrac{A}{10-A} = \dfrac{1}{4}\mathrm{e}^{\frac{t}{5}\ln\frac{8}{3}} = \dfrac{1}{4}\mathrm{e}^{\ln\left(\frac{8}{3}\right)\frac{t}{5}} = \dfrac{1}{4}\left(\dfrac{8}{3}\right)^{\frac{t}{5}}.$

$$A = \frac{10}{4}\left(\frac{8}{3}\right)^{\frac{t}{5}} - \frac{A}{4}\left(\frac{8}{3}\right)^{\frac{t}{5}}.$$

$$A\left(1 + \frac{1}{4}\left(\frac{8}{3}\right)^{\frac{t}{5}}\right) = \frac{5}{2}\left(\frac{8}{3}\right)^{\frac{t}{5}}.$$

$$A = \frac{1 + \frac{1}{4}\left(\frac{8}{3}\right)^{\frac{t}{5}}}{\frac{5}{2}\left(\frac{8}{3}\right)^{\frac{t}{5}}}$$

$$= \frac{10\left(\frac{8}{3}\right)^{\frac{t}{5}}}{4 + \left(\frac{8}{3}\right)^{\frac{t}{5}}}.$$

14