2024 JPJC J2 H2 Prelims Paper 1 Solutions

1

Let $v_n = an^3 + bn^2 + cn + d$

$v_1 = -9 : a+b+c+d = -9 - - - (1)$

$v_2 = 7 : 8a+4b+2c+d = 7 - - - (2)$

$v_3 = 47 : 27a+9b+3c+d = 47 - - - (3)$

$v_4 = 141 : 64a+16b+4c+d = 141 - - - (4)$

Use GC: $a = 5, b = -18, c = 35, d = -31$

$v_n = 5n^3 - 18n^2 + 35n - 31$

2

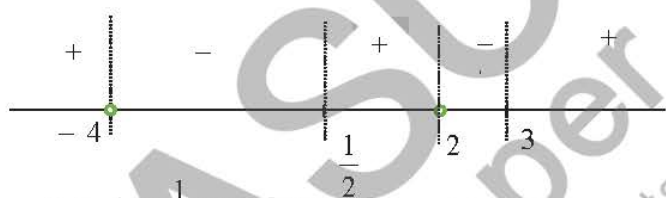
$$\frac{11x-19}{x^2+2x-8} \le 2$$

$$\frac{11x-19}{x^2+2x-8} - 2 \le 0$$

$$\frac{11x-19-2x^2-4x+16}{x^2+2x-8} \le 0$$

$$\frac{-2x^2+7x-3}{x^2+2x-8} \le 0$$

$$\frac{(2x-1)(x-3)}{(x+4)(x-2)} \ge 0$$



$x < -4$ or $\frac{1}{2} \le x < 2$ or $x \ge 3$

Replace x by e^{-x}

$e^{-x} < -4$ or $\frac{1}{2} \le e^{-x} < 2$ or $e^{-x} \ge 3$

(no solution) , $\ln\frac{1}{2} \le -x < \ln 2$ or $-x \ge \ln 3$

$-\ln 2 < x \le \ln 2$ or $x \le -\ln 3$

3

$y = 5\sin\theta \Rightarrow \frac{dy}{d\theta} = 5\cos\theta$

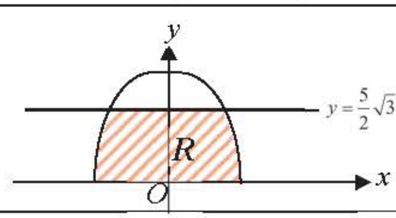
$y = 0, \quad 0 = 5\sin\theta \Rightarrow \theta = 0$

$y = \frac{5}{2}\sqrt{3}, \quad \frac{5}{2}\sqrt{3} = 5\sin\theta \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

Volume

$= \pi\int_0^{\frac{5\sqrt{3}}{2}} x^2 \, dy$

$= \pi\int_0^{\frac{5\sqrt{3}}{2}} \sqrt{25-y^2} \, dy$



$$= \pi \int_0^{\frac{\pi}{3}} \sqrt{25 - 25\sin^2\theta}\,(5\cos\theta)\,d\theta$$

$$= 5\pi \int_0^{\frac{\pi}{3}} \sqrt{25(1-\sin^2\theta)}\,\cos\theta\,d\theta$$

$$= 25\pi \int_0^{\frac{\pi}{3}} \sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$$= 25\pi \int_0^{\frac{\pi}{3}} \cos^2\theta\,d\theta$$

$$= \frac{25}{2}\pi \int_0^{\frac{\pi}{3}} 1+\cos 2\theta\,d\theta$$

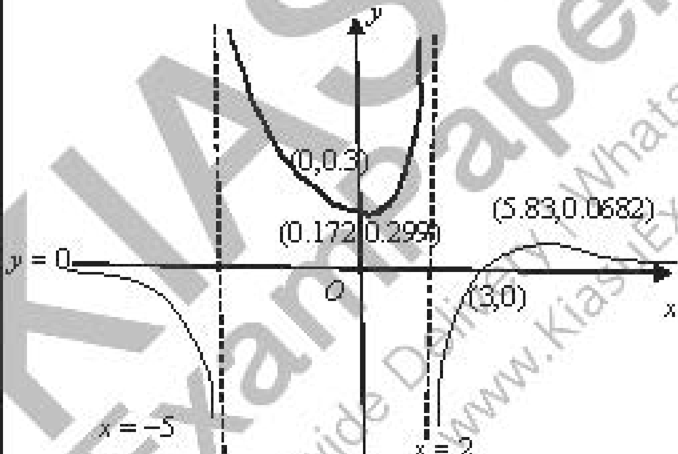
$$= \frac{25}{2}\pi \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{25}{2}\pi \left[\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right]$$

$$= \frac{25}{2}\pi \left[\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right]$$

$$= \frac{25}{6}\pi^2 + \frac{25}{8}\sqrt{3}\pi \quad \text{(exact)}$$

4(i)



(ii)

$$y = f(x) = \frac{x-3}{(x-2)(x+5)}$$

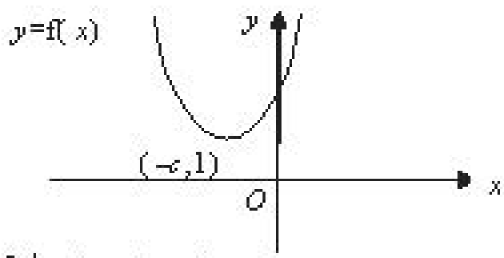
$$\rightarrow\ y = f(2x) = \frac{2x-3}{(2x-2)(2x+5)}$$

$$\rightarrow y = (2)f(2x) = (2)\frac{2x-3}{(2x-2)(2x+5)}\ y = \frac{2x-3}{(x-1)(2x+5)}$$

Scaling parallel to the x-axis by scale factor $\frac{1}{2}$.

Scaling parallel to the y-axis by scale factor 2.

5(i)



f^{-1} exists for $x \le -c$

$k = -c$

(ii)

$y = \mathrm{e}^{(x+c)^2}$

$(x+c)^2 = \ln(y)$

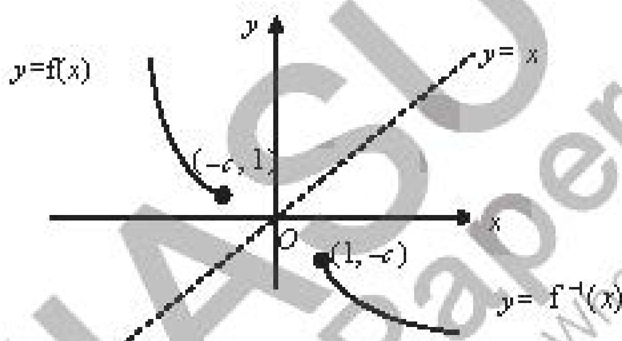
$x + c = \pm\sqrt{\ln(y)}$

$x = -c \pm \sqrt{\ln(y)}$

Since $x \le -c$, $\quad x = -c - \sqrt{\ln(y)}$

$\mathrm{f}^{-1}(x) = -c - \sqrt{\ln x},\ x \ge 1$

(iii)



Graphs of f and f^{-1}
are reflections of each other about the line $y = x$

(iv)

Range of f = $[1, \infty)$ and Domain of g = $(0, \infty)$

Range of f $\subseteq$ Domain of g

Hence gf exists

gf(x)= $\ln \mathrm{e}^{(x+c)^2}$

$\quad = (x+c)^2,\ x \le -c$

6(i)

$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{AC} = (m\underset{\sim}{a} + n\underset{\sim}{b}) - \underset{\sim}{a} = (m-1)\underset{\sim}{a} + n\underset{\sim}{b}$

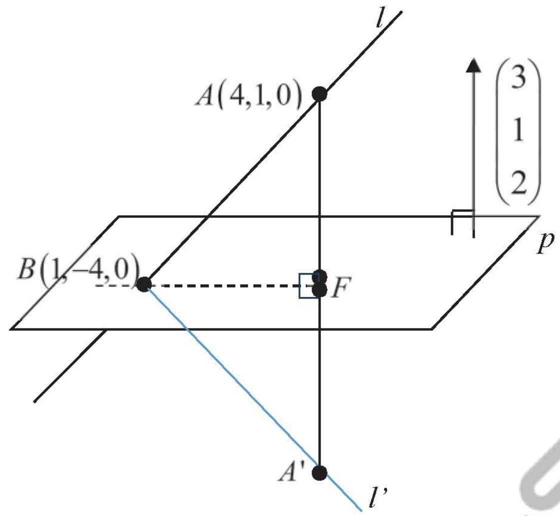
Area of triangle ABC

$= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$= \frac{1}{2}\left|(\underset{\sim}{b} - \underset{\sim}{a}) \times ((m-1)\underset{\sim}{a} + n\underset{\sim}{b})\right|$

	$=\frac{1}{2}\left\|(m-1)(\underset{\sim}{b}\times\underset{\sim}{a})+n(\underset{\sim}{b}\times\underset{\sim}{b})-(m-1)(\underset{\sim}{a}\times\underset{\sim}{a})-n(\underset{\sim}{a}\times\underset{\sim}{b})\right\|$ $=\frac{1}{2}\left\|(m-1)(\underset{\sim}{b}\times\underset{\sim}{a})+\underset{\sim}{0}+\underset{\sim}{0}+n(\underset{\sim}{b}\times\underset{\sim}{a})\right\|$ $=\frac{1}{2}(m+n-1)\left\|\underset{\sim}{b}\times\underset{\sim}{a}\right\|$ $=\frac{1}{2}(m+n-1)\left\|\underset{\sim}{b}\right\|\left\|\underset{\sim}{a}\right\|\sin 30^\circ$ $=\frac{1}{2}(m+n-1)\left\|\underset{\sim}{b}\right\|\left(4\left\|\underset{\sim}{b}\right\|\right)\left(\frac{1}{2}\right)$ $=(m+n-1)\left\|\underset{\sim}{b}\right\|^2$
(ii)	$\overrightarrow{OD}=\frac{1}{2}\underset{\sim}{a}$ $\quad\overrightarrow{OE}=\frac{3}{5}\underset{\sim}{b}$
(iii)	$l_{BD}:\underset{\sim}{r}=\underset{\sim}{b}+\lambda_1\overrightarrow{BD}$ $=\underset{\sim}{b}+\lambda_1(\underset{\sim}{d}-\underset{\sim}{b})$ $=\underset{\sim}{b}+\lambda_1\left(\frac{1}{2}\underset{\sim}{a}-\underset{\sim}{b}\right)$ $=\underset{\sim}{b}+\lambda(\underset{\sim}{a}-2\underset{\sim}{b})$, where $\lambda=\frac{\lambda_1}{2}$ $=\lambda\underset{\sim}{a}+(1-2\lambda)\underset{\sim}{b}$ (shown) $l_{AE}:\underset{\sim}{r}=\underset{\sim}{a}+\mu_1\overrightarrow{AE}$ $=\underset{\sim}{a}+\mu_1(\underset{\sim}{e}-\underset{\sim}{a})$ $=\underset{\sim}{a}+\mu_1\left(\frac{3}{5}\underset{\sim}{b}-\underset{\sim}{a}\right)$ $=\underset{\sim}{a}+\mu(3\underset{\sim}{b}-5\underset{\sim}{a})$, where $\mu=\frac{\mu_1}{5}$ $=(1-5\mu)\underset{\sim}{a}+3\mu\underset{\sim}{b}$ Since lines BD and AE meet, $\lambda\underset{\sim}{a}+(1-2\lambda)\underset{\sim}{b}=(1-5\mu)\underset{\sim}{a}+3\mu\underset{\sim}{b}$ Comparing coefficients of $\underset{\sim}{a}$, $\lambda=1-5\mu\Rightarrow\lambda+5\mu=1$ --- (1) Comparing coefficients of $\underset{\sim}{b}$, $1-2\lambda=3\mu\Rightarrow 2\lambda+3\mu=1$ --- (2) Using GC, $\lambda=\frac{2}{7}$, $\mu=\frac{1}{7}$ $\overrightarrow{OF}=\frac{2}{7}\underset{\sim}{a}+\left[1-2\left(\frac{2}{7}\right)\right]\underset{\sim}{b}$ $=\frac{2}{7}\underset{\sim}{a}+\frac{3}{7}\underset{\sim}{b}$

7(a)	$l: \underset{\sim}{r} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}, t \in \mathbb{R} \quad --(1)$ For plane p, $\underset{\sim}{n} = \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} \times \begin{pmatrix} -3 \\ a \\ 1 \end{pmatrix} = \begin{pmatrix} -5-a \\ -4 \\ a-15 \end{pmatrix}$ Since l and p do not meet in a unique point, $\begin{pmatrix} -5-a \\ -4 \\ a-15 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} = 0$ $3(-5-a)-20=0$ $-3a=35$ $a=-\frac{35}{3}$
(b)(i)	Given $a=7$, $\underset{\sim}{n} = \begin{pmatrix} -5-7 \\ -4 \\ 7-15 \end{pmatrix} = -4\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ $p: \underset{\sim}{r} \bullet \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = -1 \text{ --- } (2)$ Subst. (1) into (2): $\left[\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}\right] \bullet \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = -1$ $13+14t=-1$ $t=-1$ Position vector of the point of intersection, $B = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}$ $\therefore B(1,-4,0)$

(ii)	 To find F, the foot of perpendicular from A to p: $l_{AF}: \ \underset{\sim}{r} = \begin{pmatrix}4\\1\\0\end{pmatrix} + s\begin{pmatrix}3\\1\\2\end{pmatrix}, s \in \mathbb{R}$ $\left[\begin{pmatrix}4\\1\\0\end{pmatrix} + s\begin{pmatrix}3\\1\\2\end{pmatrix}\right] \bullet \begin{pmatrix}3\\1\\2\end{pmatrix} = -1$ $13 + 14s = -1$ $s = -1$ $\overrightarrow{OF} = \begin{pmatrix}4\\1\\0\end{pmatrix} - \begin{pmatrix}3\\1\\2\end{pmatrix} = \begin{pmatrix}1\\0\\-2\end{pmatrix}$ Since F is the mid-point of AA', $\overrightarrow{OF} = \dfrac{\overrightarrow{OA} + \overrightarrow{OA}'}{2}$ $\overrightarrow{OA}' = 2\overrightarrow{OF} - \overrightarrow{OA}$ $= 2\begin{pmatrix}1\\0\\-2\end{pmatrix} - \begin{pmatrix}4\\1\\0\end{pmatrix} = \begin{pmatrix}-2\\-1\\-4\end{pmatrix}$ $\overrightarrow{BA}' = \begin{pmatrix}-2\\-1\\-4\end{pmatrix} - \begin{pmatrix}1\\-4\\0\end{pmatrix} = \begin{pmatrix}-3\\3\\-4\end{pmatrix}$ $l': \ \underset{\sim}{r} = \begin{pmatrix}1\\-4\\0\end{pmatrix} + \alpha\begin{pmatrix}-3\\3\\-4\end{pmatrix}, \alpha \in \mathbb{R}$ $\dfrac{1-x}{3} = \dfrac{y+4}{3} = -\dfrac{z}{4}$

8 (i)

Month	Amount owed at beginning of the month	Amount owed at the end of the month
1	$200000(1.005)$	$200000(1.005) - x$
2	$200000(1.005)^2 - (1.005)x$	$200000(1.005)^2 - (1.005)x - x$
3	$200000(1.005)^3 - (1.005)^2x - (1.005)x$	$200000(1.005)^3 - (1.005)^2x - (1.005)x - x$

Amount owed at the end of n months

$$= 200000(1.005)^n - (1.005)^{n-1}x - (1.005)^{n-2}x - \ldots - 1.005x - x$$

$$= 200000(1.005)^n - x\left[1 + 1.005 + \ldots + (1.005)^{n-2} + (1.005)^{n-1}\right]$$

$$= 200000(1.005)^n - \frac{x\left[1 - 1.005^n\right]}{1 - 1.005}$$

$$= 200000(1.005)^n - 200x\left[1.005^n - 1\right]$$

(ii)

$$200000(1.005)^n - 200x\left[1.005^n - 1\right] \le 0$$

$$200000(1.005)^n - 200(1500)\left[1.005^n - 1\right] \le 0$$

$$300000 - 100000(1.005)^n \le 0$$

$$(1.005)^n \ge 3$$

$$n \ge \frac{\ln 3}{\ln 1.005}$$

$$n \ge 220.27$$

Alternatively,
Use GC table

n	$200000(1.005)^n - 200(1500)\left[1.005^n - 1\right]$
219	1896.19
220	$405.67 > 0$
221	$-1092.30 < 0$

$n = 221$

At the end of 220 months, Selena owed
$200000(1.005)^{220} - 200(1500)\left[1.005^{220} - 1\right] = \$405.67$

Last repayment amount to be repaid on the 221[st] month
$= \$405.67 \times 1.005 = \$407.70$ (2d.p.)

221 months = 18 years 5 months
Full repayment on: 31 May 2043

(iii)

$$200000(1.005)^{120} - 200x\left[1.005^{120} - 1\right] \le 0$$

$$363879.3468 - 163.8793468x \le 0$$

$$x \ge 2220.410039$$

\$2220.42 (2d.p.) [\$2220.41 not accepted]

9(a)

Sub. $z=1+2i$ into $z^4-z^3-9z^2+sz+t=0$

$$(1+2i)^4-(1+2i)^3-9(1+2i)^2+s(1+2i)+t=0$$

$$(-7-24i)-(-11-2i)-9(-3+4i)+s(1+2i)+t=0$$

$$(31+s+t)+(2s-58)i=0$$

Comparing imaginary parts,

$2s-58=0$

$\underline{\underline{s=29}}$

Comparing real parts,

$31+s+t=0$

$t=-31-s$

$=\underline{\underline{-60}}$

Now $z^4-z^3-9z^2+29z-60=0$

Using GC the other roots are $\underline{1-2i, 3, -4}$.

Alternative solution

Since $z^4-z^3-9z^2+sz+t=0$ is a polynomial equation with real coefficients and $1+2i$ is a root, $1-2i$ is another root.

Quadratic factor $=\left[z-(1+2i)\right]\left[z-(1-2i)\right]$

$=\left[(z-1)-2i\right]\left[(z-1)+2i\right]$

$=(z-1)^2-(2i)^2$

$=z^2-2z+5$

Let $z^4-z^3-9z^2+sz+t=(z^2-2z+5)(z^2+az+b)$.

By comparing coefficients,

z^3: $-1=a-2 \Rightarrow a=1$

z^2: $-9=b-2a+5 \Rightarrow b=-12$

z: $s=-2b+5a=29$

constant term: $t=5b=-60$

Now $z^4-z^3-9z^2+29z-60=0$

Using GC (polyroot finder), the other roots are $\underline{1-2i, 3, -4}$.

(b)

$$\arg\left(\frac{w^3}{iw^*}\right)=\arg w^3-\arg(iw^*)$$

$$=3\arg w-(\arg i+\arg w^*)$$

$$=3\arg w-\left(\frac{\pi}{2}-\arg w\right)$$

$$=4\arg w-\frac{\pi}{2}$$

	For $\frac{w^3}{iw^*}$ to be purely imaginary, $\arg\left(\frac{w^3}{iw^*}\right)=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},\ldots$ $4\arg(w)-\frac{\pi}{2}=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},\ldots$ $4\arg(w)=0,\ \pi,\ 2\pi,\ -\pi,\ 3\pi,-2\pi$ $\arg(w)=\frac{\pi}{4},0,\ \frac{\pi}{2},\frac{-\pi}{4},\frac{3\pi}{4},-\frac{\pi}{2}$ Since $w=a+ib$ and a and b are positive real numbers, $0<\arg(w)<\frac{\pi}{2}$. $\therefore \arg(w)=\frac{\pi}{4}$ $\tan^{-1}\left(\frac{b}{a}\right)=\frac{\pi}{4}$ $\frac{b}{a}=1$ $b=a$ $w=a+ia$

10(i)	$\frac{dv}{dt}=2e^{-0.1t}$ $v=\int 2e^{-0.1t}\,dt$ $v=2\left(\frac{e^{-0.1t}}{-0.1}\right)+c$ $v=-20e^{-0.1t}+c$ When $t=0$, $v=0$ $0=-20e^{0}+c$ $c=20$ $v=20-20e^{-0.1t}$ Subt $v=10$ $10=20-20e^{-0.1t}$ $20e^{-0.1t}=10$ $e^{-0.1t}=\frac{1}{2}$ $-0.1t=\ln\frac{1}{2}$ $t=-10\ln\frac{1}{2}=10\ln 2$ (exact)
(ii)	As $t\to\infty, e^{-0.1t}\to 0, v\to 20$ Eventually, the speed increases and tend to 20 ms^{-1}

(iii)

$$-2\frac{\mathrm{d}w}{\mathrm{d}t}=(w-3)(w+2)$$

$$\frac{1}{(w-3)(w+2)}\frac{\mathrm{d}w}{\mathrm{d}t}=-\frac{1}{2}$$

$$\int\frac{1}{(w-3)(w+2)}\,\mathrm{d}w=\int-\frac{1}{2}\,\mathrm{d}t$$

<u>Method 1: Partial fractions</u>

$$\frac{1}{(w-3)(w+2)}=\frac{A}{(w-3)}+\frac{B}{(w+2)}$$

$$1=A(w+2)+B(w-3)$$

$$1=-5B\Rightarrow B=-\frac{1}{5}$$

$$1=5A\Rightarrow A=\frac{1}{5}$$

$$\frac{1}{5}\int\frac{1}{(w-3)}-\frac{1}{(w+2)}\,\mathrm{d}w=\int-\frac{1}{2}\,\mathrm{d}t$$

<u>Method 2: Completing the square</u>

$$\int\frac{1}{w^2-w-6}\,\mathrm{d}w=\int-\frac{1}{2}\,\mathrm{d}t$$

$$\int\frac{1}{\left(w-\frac{1}{2}\right)^2-\left(\frac{5}{2}\right)^2}\,\mathrm{d}w=\int-\frac{1}{2}\,\mathrm{d}t$$

$$\frac{1}{2\left(\frac{5}{2}\right)}\ln\left|\frac{w-\frac{1}{2}-\frac{5}{2}}{w-\frac{1}{2}+\frac{5}{2}}\right|=\int-\frac{1}{2}\,\mathrm{d}t$$

$$\frac{1}{5}\left[\ln|w-3|-\ln|w+2|\right]=-\frac{1}{2}t+c$$

$$\frac{1}{5}\ln\left|\frac{w-3}{w+2}\right|=-\frac{1}{2}t+c$$

$$\ln\left|\frac{w-3}{w+2}\right|=-\frac{5}{2}t+5c$$

$$\left|\frac{w-3}{w+2}\right|=\mathrm{e}^{-\frac{5}{2}t+5c}$$

$$\frac{w-3}{w+2}=\pm\mathrm{e}^{-\frac{5}{2}t+5c}$$

$$\frac{w-3}{w+2}=A\mathrm{e}^{-\frac{5}{2}t},\quad A=\pm\mathrm{e}^{5c}$$

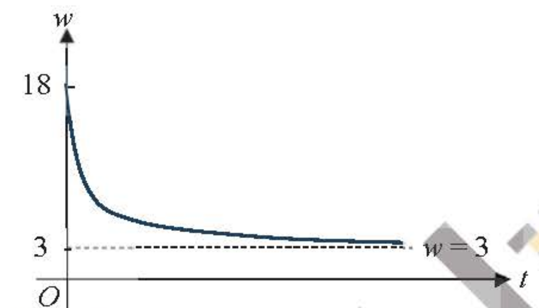
Sub w = 18,

$$\frac{18-3}{18+2}=A\mathrm{e}^{0}$$

$$A=\frac{15}{20}=\frac{3}{4}$$

$$\frac{w-3}{w+2}=\frac{3}{4}\mathrm{e}^{-\frac{5}{2}t}$$

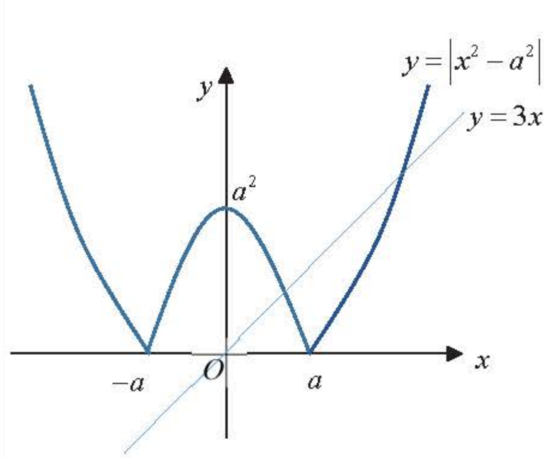
$$w-3=\frac{3}{4}w\mathrm{e}^{-\frac{5}{2}t}+\frac{3}{2}\mathrm{e}^{-\frac{5}{2}t}$$

	$w-\frac{3}{4}we^{-\frac{5}{2}t}=3+\frac{3}{2}e^{-\frac{5}{2}t}$ $w=\frac{3+\frac{3}{2}e^{-\frac{5}{2}t}}{1-\frac{3}{4}e^{-\frac{5}{2}t}}$ $w=\frac{12+6e^{-\frac{5}{2}t}}{4-3e^{-\frac{5}{2}t}}$
(iv)	The speed will not fall below 3 ms^{-1}.

11(i)	$y=x\tan\theta-\frac{10x^2}{2u^2\cos^2\theta}$ $-1.8=15\tan\left(\frac{\pi}{4}\right)-\frac{10(15)^2}{2u^2\cos^2\left(\frac{\pi}{4}\right)}$ $-1.8=15(1)-\frac{10(15)^2}{2u^2\left(\frac{1}{2}\right)}$ $\frac{10(15)^2}{u^2}=15+1.8$ $u^2=\frac{10(15)^2}{16.8}$ $u=11.6$
(ii)	$y=x\tan\theta-\frac{10x^2}{2(10)^2\cos^2\theta}$ When $y=-1.8$ $\therefore\quad -1.8=x\tan\theta-\frac{x^2}{20\cos^2\theta}$ $-36\cos^2\theta=20x\tan\theta\cos^2\theta-x^2$ $x^2-20x\sin\theta\cos\theta-36\cos^2\theta=0$ $x^2-10x\sin 2\theta-18(1+\cos 2\theta)=0$ $x^2-10x\sin 2\theta-18\cos 2\theta-18=0$ (Shown)

(iii)	Differentiate w.r.t. θ, we have $2x\frac{dx}{d\theta}-10\frac{dx}{d\theta}\sin 2\theta-20x\cos 2\theta+36\sin 2\theta=0$ At stationary value of x, $\frac{dx}{d\theta}=0$ $-20x\cos 2\theta+36\sin 2\theta=0$ $36\sin 2\theta=20x\cos 2\theta$ $x=\frac{36\sin 2\theta}{20\cos 2\theta}$ $x=\frac{9}{5}\tan 2\theta$
(iv)	Sub into equation, we have $\left(\frac{9}{5}\tan 2\theta\right)^2-10\left(\frac{9}{5}\tan 2\theta\right)\sin 2\theta-18\cos 2\theta-18=0$ $\frac{81}{25}\tan^2 2\theta-18\tan 2\theta\sin 2\theta-18\cos 2\theta-18=0$ $81\tan^2 2\theta-450\sin 2\theta\tan 2\theta-450\cos 2\theta-450=0$ Using GC, $\theta=0.70883\approx 0.71$ (2 decimal places), $\frac{\pi}{2}$ (reject) Therefore, stationary value of $x=\frac{9}{5}\tan 2(0.70883)=11.7$

1((a)



(b)

At point of intersection where $0 < x < a$,

$$-(x^2-a^2)=3x$$

$$x^2-a^2=-3x$$

$$x^2+3x-a^2=0$$

$$x=\frac{-3\pm\sqrt{9+4a^2}}{2}$$

$$x=\frac{-3+\sqrt{9+4a^2}}{2} \quad \text{(since } x>0\text{)}$$

2(a)

$$\int \sin px \cos qx \, \mathrm{d}x$$

$$=\frac{1}{2}\int 2\sin px\cos qx \, \mathrm{d}x$$

$$=\frac{1}{2}\int \sin(p+q)x+\sin(p-q)x \, \mathrm{d}x$$

$$=\frac{1}{2}\left[-\frac{1}{p+q}\cos(p+q)x-\frac{1}{p-q}\cos(p-q)x\right]+C$$

<u>Alternatively,</u>

$$\int \sin px \cos qx \, \mathrm{d}x$$

$$=\frac{1}{2}\int 2\cos qx\sin px \, \mathrm{d}x$$

$$=\frac{1}{2}\int \sin(q+p)x-\sin(q-p)x \, \mathrm{d}x$$

$$=\frac{1}{2}\left[-\frac{1}{q+p}\cos(q+p)x+\frac{1}{q-p}\cos(q-p)x\right]+C$$

(b)

$$\int x\sin mx \, \mathrm{d}x=-\frac{1}{m}x\cos mx+\int\frac{1}{m}\cos mx \, \mathrm{d}x$$

$$=-\frac{1}{m}x\cos mx+\frac{1}{m^2}\sin mx+C$$

$u=x \quad \frac{\mathrm{d}v}{\mathrm{d}x}=\sin mx$

$\frac{\mathrm{d}u}{\mathrm{d}x}=1 \quad v=-\frac{\cos mx}{m}$

(c)

$\int_0^{\pi} x \sin mx \, dx$

$= \left[-\frac{1}{m} x \cos mx + \frac{1}{m^2} \sin mx \right]_0^{\pi}$

$= \left[-\frac{1}{m} \pi \cos m\pi + \frac{1}{m^2} \sin m\pi \right] - \left[-\frac{1}{m}(0)\cos 0 + \frac{1}{m^2}\sin(0) \right]$

$= -\frac{1}{m} \pi \cos m\pi$

When m is odd, $\int_0^{\pi} x \sin mx \, dx = -\frac{1}{m}\pi(-1) = \frac{1}{m}\pi$

When m is even, $\int_0^{\pi} x \sin mx \, dx = -\frac{1}{m}\pi(1) = -\frac{1}{m}\pi$

$k = 1$ or -1

3(i)

$f'(x) = e^{\tan^{-1} x}\left(\frac{1}{1+x^2}\right)$

$(1+x^2)f'(x) = f(x)$

<u>Alternatively,</u>

$f(x) = e^{\tan^{-1} x}$

$\ln f(x) = \tan^{-1} x$

Differentiate with respect to x

$\frac{1}{f(x)} f'(x) = \frac{1}{1+x^2}$

$(1+x^2)f'(x) = f(x)$

(ii)

$(1+x^2)f'(x) = f(x)$

Differentiate with respect to x

$(1+x^2)f''(x) + 2xf'(x) = f'(x)$

Differentiate with respect to x

$(1+x^2)f'''(x) + 2xf''(x) + 2xf''(x) + 2f'(x) = f''(x)$

$(1+x^2)f'''(x) + 4xf''(x) + 2f'(x) = f''(x)$

When $x = 0$, $y = f(0) = e^{\tan^{-1} 0} = 1$

$(1+0)f'(x) = 1 \Rightarrow f'(x) = 1$

$(1+0)f''(x) + 0 = 1 \Rightarrow f''(x) = 1$

$(1+0)f'''(x) + 0 + 2(1) = 1 \Rightarrow f'''(x) = -1$

$y = f(x) = e^{\tan^{-1} x} = 1 + x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots \quad \approx 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3$

(iii)

$\int_0^{0.5} f(x)\,dx \approx \int_0^{0.5} 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \, dx \approx 0.6432$ (4s.f.)

(iv)

The series is a good approximation for $f(x)$ if x is close to 0. Since <u>$x = 1$ is not close to 0,</u> it is <u>not suitable</u> to be used to estimate $e^{\frac{\pi}{4}}$.

4(i)	$\dfrac{7}{n-2}-\dfrac{5}{n-1}-\dfrac{2}{n}=\dfrac{7(n-1)n-5(n-2)n-2(n-2)(n-1)}{(n-2)(n-1)n}$ $=\dfrac{7n^2-7n-5n^2+10n-2n^2+6n-4}{(n-2)(n-1)n}$ Obtain $\dfrac{9n-4}{(n-2)(n-1)n}$ (Shown) <u>Alternative : By partial fractions</u> Let $\dfrac{9n-4}{(n-2)(n-1)n}=\dfrac{A}{n-2}+\dfrac{B}{n-1}+\dfrac{C}{n}$ $9n-4=A(n-1)n+B(n-2)n+C(n-2)(n-1)$ Subst $n=2$, obtain $A=7$ Subst $n=1$, obtain $B=-5$ Subst $n=0$, obtain $C=-2$ Obtain $\dfrac{9n-4}{(n-2)(n-1)n}$ (Shown)
(ii)	$\displaystyle\sum_{n=3}^{N}\frac{9n-4}{(n-2)(n-1)n}=\sum_{n=3}^{N}\left[\frac{7}{(n-2)}-\frac{5}{(n-1)}-\frac{2}{n}\right]$ $=\left[\begin{array}{l}\frac{7}{1}-\frac{5}{2}-\frac{2}{3}\\+\frac{7}{2}-\frac{5}{3}-\frac{2}{4}\\+\frac{7}{3}-\frac{5}{4}-\frac{2}{5}\\+\frac{7}{4}-\frac{5}{5}-\frac{2}{6}\\+\ldots\ldots\\+\frac{7}{N-4}-\frac{5}{N-3}-\frac{2}{N-2}\\+\frac{7}{N-3}-\frac{5}{N-2}-\frac{2}{N-1}\\+\frac{7}{N-2}-\frac{5}{N-1}-\frac{2}{N}\end{array}\right]$ $=8-\left[\dfrac{7}{N-1}+\dfrac{2}{N}\right]$
(iii)	$\displaystyle\sum_{n=3}^{N}\frac{9n-4}{(n-2)(n-1)n}=8-\left[\frac{7}{N-1}+\frac{2}{N}\right]$ As $N\to\infty$, $\dfrac{7}{N-1}\to 0$ and $\dfrac{2}{N}\to 0$, $\displaystyle\sum_{n=3}^{N}\frac{9n-4}{(n-2)(n-1)n}\to 8$ which is finite. Hence $\displaystyle\sum_{n=3}^{N}\frac{9n-4}{(n-2)(n-1)n}$ is convergent and the sum to infinity is 8.

(iv)

$$\sum_{n=3}^{N}\frac{9n-4}{(n-2)(n-1)n}=\frac{23}{(1)(2)(3)}+\left(\frac{32}{(2)(3)(4)}+\frac{41}{(3)(4)(5)}+\ldots\ldots\ldots\frac{9N-4}{(N-2)(N-1)N}\right)$$

$$\sum_{n=2}^{N}\frac{9n+14}{n(n+1)(n+2)}$$

$$=\left(\frac{32}{(2)(3)(4)}+\frac{41}{(3)(4)(5)}+\ldots+\frac{9N-4}{(N-2)(N-1)(N)}\right)$$

$$+\frac{9N+5}{(N-1)(N)(N+1)}+\frac{9N+14}{N(N+1)(N+2)}$$

$$=\sum_{n=4}^{N+2}\frac{9n-4}{(n-2)(n-1)n}\quad \text{Note : replace } n \text{ by } n-2$$

$$=\sum_{n=3}^{N+2}\frac{9n-4}{(n-2)(n-1)n}-\frac{23}{(1)(2)(3)}$$

$$=8-\left[\frac{7}{N+2-1}+\frac{2}{N+2}\right]-\frac{23}{6}$$

$$=\frac{25}{6}-\left[\frac{7}{N+1}+\frac{2}{N+2}\right]$$

5(i)

$$\overrightarrow{OA}=\begin{pmatrix}5\\0\\0\end{pmatrix},\ \overrightarrow{OB}=\begin{pmatrix}-4\\-3\\0\end{pmatrix},\ \overrightarrow{OC}=\begin{pmatrix}-5\\0\\6\end{pmatrix}$$

Since $ABCD$ is a parallelogram,

$$\overrightarrow{AD}=\overrightarrow{BC}$$

$$\overrightarrow{OD}-\overrightarrow{OA}=\overrightarrow{OC}-\overrightarrow{OB}$$

$$=\begin{pmatrix}-5\\0\\6\end{pmatrix}-\begin{pmatrix}-4\\-3\\0\end{pmatrix}=\begin{pmatrix}-1\\3\\6\end{pmatrix}$$

$$\overrightarrow{OD}=\begin{pmatrix}-1\\3\\6\end{pmatrix}+\begin{pmatrix}5\\0\\0\end{pmatrix}$$

$$=\begin{pmatrix}4\\3\\6\end{pmatrix}$$

	$\overrightarrow{BA}=\begin{pmatrix}5\\0\\0\end{pmatrix}-\begin{pmatrix}-4\\-3\\0\end{pmatrix}=\begin{pmatrix}9\\3\\0\end{pmatrix}$ $\overrightarrow{BA}\cdot\overrightarrow{BC}=\begin{pmatrix}9\\3\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\6\end{pmatrix}=0$ Since $ABCD$ is a parallelogram and $AB\perp BC$, $ABCD$ is a rectangle.
(ii)	$\overrightarrow{BA}\times\overrightarrow{BC}=\begin{pmatrix}9\\3\\0\end{pmatrix}\times\begin{pmatrix}-1\\3\\6\end{pmatrix}=3\begin{pmatrix}3\\1\\0\end{pmatrix}\times\begin{pmatrix}-1\\3\\6\end{pmatrix}=6\begin{pmatrix}3\\-9\\5\end{pmatrix}$ Normal to the plane $ABC=\underset{\sim}{n}=\begin{pmatrix}3\\-9\\5\end{pmatrix}$. $\overrightarrow{OA}\cdot\begin{pmatrix}3\\-9\\5\end{pmatrix}=\begin{pmatrix}5\\0\\0\end{pmatrix}\cdot\begin{pmatrix}3\\-9\\5\end{pmatrix}=15$ Vector equation of plane ABC is $\underset{\sim}{r}\cdot\begin{pmatrix}3\\-9\\5\end{pmatrix}=15$ Cartesian equation of plane ABC is $\underline{3x-9y+5z=15}$.
(iii)	Normal vector of the base is parallel to $\overrightarrow{OE}$. Hence, normal vector of the base $=\begin{pmatrix}0\\0\\1\end{pmatrix}$ Acute angle between plane ABC and the base $=\cos^{-1}\dfrac{\left\lvert\begin{pmatrix}3\\-9\\5\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}3\\-9\\5\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}$ $=\cos^{-1}\dfrac{\lvert 5\rvert}{\sqrt{3^2+9^2+5^2}\sqrt{1}}$ $=\cos^{-1}\dfrac{5}{\sqrt{115}}$ $=\underline{62.2^\circ}$ (1 dp)

(iv)

Since $AF : AE = 1:5$,

$$\overrightarrow{AF} = \frac{1}{5}\overrightarrow{AE} = \frac{1}{5}\left[\begin{pmatrix}0\\0\\10\end{pmatrix} - \begin{pmatrix}5\\0\\0\end{pmatrix}\right] = \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

Length of projection of AF onto plane ABC

$$= \frac{\left|\overrightarrow{AF} \times \underset{\sim}{n}\right|}{\left|\underset{\sim}{n}\right|}$$

$$= \frac{\left|\begin{pmatrix}-1\\0\\2\end{pmatrix} \times \begin{pmatrix}3\\-9\\5\end{pmatrix}\right|}{\left|\begin{pmatrix}3\\-9\\5\end{pmatrix}\right|}$$

$$= \frac{\left|\begin{pmatrix}18\\11\\9\end{pmatrix}\right|}{\sqrt{3^2 + (-9)^2 + 5^2}}$$

$$= \frac{\sqrt{526}}{\sqrt{115}}$$

$$= 2.1387$$

$$\approx 2.14$$

6(a)(i)

Since there is only one way the letters are in alphabetical order,

Total number of ways $= \dfrac{11!}{3!2!2!} - 1$

$= 1\ 663\ 199$

(a)(ii)

$\uparrow$ $\boxed{\text{A A O I E}}$ $\uparrow$ R $\uparrow$ R $\uparrow$ T $\uparrow$

$\boxed{\text{PP}}$ P

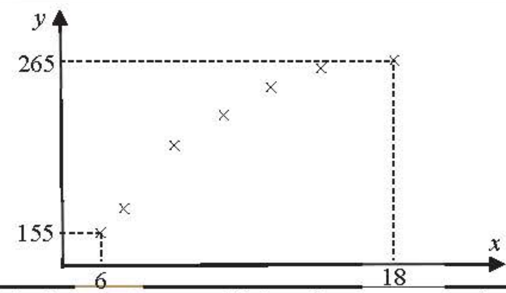
Total number of ways $= \dfrac{4!}{2!} \times \dfrac{5!}{2!} \times {}^5P_2 = 14\ 400$

(b)

<u>Method 1</u>: Probability $= \dfrac{6}{11} \times \dfrac{5}{10} \times \dfrac{4}{9} \times \dfrac{3!}{2!} + \dfrac{5}{11} \times \dfrac{4}{10} \times \dfrac{3}{9} = \dfrac{14}{33}$ (or 0.424)

<u>Method 2</u>: Probability $= \dfrac{{}^6C_1\,{}^5C_2 + {}^5C_3}{{}^{11}C_3}$

<u>Method 3</u>: Probability $= 1 - \dfrac{{}^6C_3}{{}^{11}C_3} - \dfrac{{}^5C_1\,{}^6C_2}{{}^{11}C_3} = \dfrac{14}{33}$

7(i)	
(ii)	**(a)** Correlation coefficient between x and y is $r = 0.9483$ (4 d.p.) **(b)** Correlation coefficient between $\ln x$ and y is $r = 0.9849$ (4 d.p.)
(iii)	From the scatter diagram, as x increases, y increases at a decreasing rate. In addition, the product moment correlation coefficient between $\ln x$ and y, 0.9849, is closer to +1 as compared to that between x and y, 0.9483. Hence $y = c\ln x + d$ is the better model.
(iv)	Since x is the independent variable, neither the regression line of x on y nor the regression line of $\ln x$ on y should be used to estimate the value of x when $y = 200$.
(v)	Equation of regression line of y on $\ln x$ is $y = 106.5611\ln x - 31.2643$ $y = 107\ln x - 31.3$ when $y = 200$, $200 = 106.5611\ln x - 31.2643$ $x = 8.76$ (3 s.f.) Since $y = 200$ is within the given range of data, which is an interpolation, and r is close to +1, indicating a strong positive linear correlation, the estimate is reliable.

8(i)	P(packet is unsatisfactory) $= 1 - (0.99)(0.98)^2(0.96)$ $= 0.087236 \approx 0.0872$ (3 s.f.)
(ii)	$X \sim B(180, 0.087236)$ $P(5 \le X < 10) = P(X \le 9) - P(X \le 4)$ $= 0.042669 \approx 0.0427$
(iii)	$P(X > r) \le 0.12$ $1 - P(X \le r) \le 0.12$ $P(X \le r) \ge 0.88$ From GC, $P(X \le 19) = 0.8429$ (< 0.88) $P(X \le 20) = 0.8947$ (> 0.88) $P(X \le 21) = 0.9323$ (> 0.88) $\therefore$ least $r = 20$
(iv)	Each packet has an equal chance of being selected and the selection of the packets is independent of one another. This method of selection is done so that a random sample will be obtained which is free from bias and will be representative of the population.

(v)	Required probability $= (0.087236)^2 \times (0.912764)^6 \times \frac{8!}{2!6!} \times (0.087236)$ $= 0.010749 \approx 0.0107$ Alternative Method Let Y be the the number of packets (out of 8) that are unsatisfactory. $Y \sim B(8, 0.087236)$ Required probability $= P(Y=2) \times (0.087236)$ $= 0.010749 \approx 0.0107$

9(i)

Box A \ Box B	(B) 4 $\left(\frac{1}{3}\right)$	(R) 5 $\left(\frac{2}{3}\right)$
(B) 1 $\left(\frac{1}{5}\right)$	5	5
(R) 2 $\left(\frac{2}{5}\right)$	8	7
(R) 3 $\left(\frac{2}{5}\right)$	12	8

$P(X=8) = P(2 \text{ from Box A and } 4 \text{ from Box B}) + P(3 \text{ from Box A and } 5 \text{ from Box B})$

$$= \left(\frac{2}{5}\right)\left(\frac{1}{3}\right)+\left(\frac{2}{5}\right)\left(\frac{2}{3}\right)$$

$$= \frac{2}{5}$$

(ii)

$$P(X=5) = \left(\frac{1}{5}\right)\left(\frac{1}{3}\right)+\left(\frac{1}{5}\right)\left(\frac{2}{3}\right)=\frac{1}{5}$$

$$P(X=7) = \left(\frac{2}{5}\right)\left(\frac{2}{3}\right)=\frac{4}{15}$$

$$P(X=12) = \left(\frac{2}{5}\right)\left(\frac{1}{3}\right)=\frac{2}{15}$$

The probability distribution of X is

x	5	7	8	12
$P(X=x)$	$\frac{1}{5}$	$\frac{4}{15}$	$\frac{2}{5}$	$\frac{2}{15}$

(iii)

$$E(X) = 5\left(\frac{1}{5}\right)+7\left(\frac{4}{15}\right)+8\left(\frac{2}{5}\right)+12\left(\frac{2}{15}\right)=\frac{23}{3}$$

$$E(X^2) = 5^2\left(\frac{1}{5}\right)+7^2\left(\frac{4}{15}\right)+8^2\left(\frac{2}{5}\right)+12^2\left(\frac{2}{15}\right)=\frac{943}{15}$$

$$Var(X) = \frac{943}{15}-\left(\frac{23}{3}\right)^2=\frac{184}{45}$$

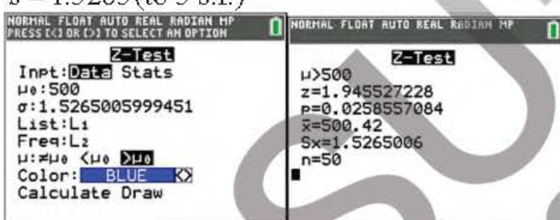
(iv)	Since sample size $n = 50$ is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(\frac{23}{3}, \frac{184}{45(50)}\right) = \mathrm{N}\left(\frac{23}{3}, \frac{92}{1125}\right)$ approximately. $\mathrm{P}\left(7.5 < \overline{X} < 8.5\right) = 0.71821 \approx 0.718$ (3 s.f.)

10(i)

Using GC,

$\bar{x} = 500.42$ (exact)

$s = 1.5265$ (to 5 s.f.)



Let X = vol. of green tea dispensed in a bottle and

μ = population mean green tea dispensed in a bottle

$\mathrm{H}_0 : \mu = 500$

$\mathrm{H}_1 : \mu > 500$

Under H_0, since $n = 50$ is large, by Central Limit Theorem,

$$\overline{X} \sim \mathrm{N}\left(500\ ,\ \frac{1.5265^2}{50}\right) \text{ approximately}$$

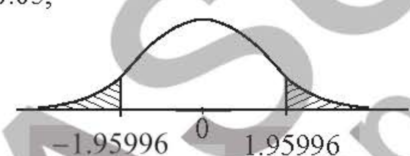
Test statistic: $z = \dfrac{\bar{x} - 500}{\left(\dfrac{1.5265}{\sqrt{50}}\right)}$

$$= \frac{500.42 - 500}{\left(\dfrac{1.5265}{\sqrt{50}}\right)} = 1.9455 \approx 1.95 \text{ (to 3 s.f)}$$

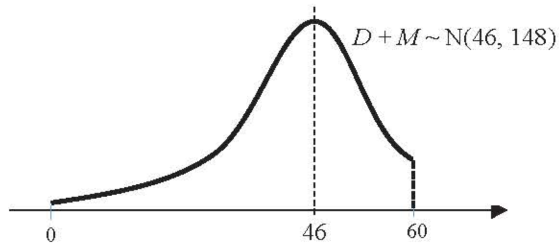
From GC, p-value $= 0.025856 \approx 0.0259$ (to 3 s.f)

Since p-value $= 0.0259 > 0.02$, we do not reject Ho at 2% level of significance and conclude that there is insufficient evidence that the production manager's suspicion is valid.

The p-value is 0.0259 and it means there is a probability of 0.0259 of observing a test statistic, $z \geq 1.95$, given that the population mean green tea dispensed in a bottle is 500ml.

(ii)	For the production manager's suspicion ($\mu > 500$) to be valid, $\mathrm{H_0}$ is rejected. Hence, p-value $= 0.025856 < \dfrac{\alpha}{100}$ $\therefore 2.59 < \alpha < 100$ (3 s.f)
(iii)	Since the sample size = 50 is large enough, Central Limit Theorem can be applied for sample means to follow a normal distribution approximately.
(b)	$s^2 = \dfrac{n}{n-1}\sigma_x{}^2 = \dfrac{50}{49}k^2$ $\mathrm{H_0} : \mu = 500$ $\mathrm{H_1} : \mu \neq 500$ Under $\mathrm{H_0}$,since $n = 50$ is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(500\ ,\ \dfrac{\left(\dfrac{50k^2}{49}\right)}{50}\right) = \mathrm{N}\left(500\ ,\ \dfrac{k^2}{49}\right)$ approximately Test statistic: $z = \dfrac{\overline{x} - 500}{\sqrt{\dfrac{k^2}{49}}} = \dfrac{502-500}{\dfrac{k}{7}} = \dfrac{14}{k}$ For $\alpha = 0.05$,  If recalibration is done accurately ($\mu = 500$), $\mathrm{H_0}$ is not rejected. Hence, $-1.95996 < z < 1.95996$ $-1.95996 < \dfrac{14}{k} < 1.95996$ $\dfrac{k}{14} < -0.51021$ (rejected) or $\dfrac{k}{14} > 0.51021$ $k > 7.1430$ $\therefore k > 7.14$ (to 3 s.f)

11(i)	Let D be the time in minutes past 7 pm that Benedict finishes his dinner. Let M be the time in minutes he spends on social media. $D \sim \mathrm{N}(10, 2^2)$ and $M \sim \mathrm{N}(k, 12^2)$, Then $D + M \sim \mathrm{N}(10 + k, 148)$ Given $\mathrm{P}(D + M > 60) = 0.125$ $\mathrm{P}\left(Z > \dfrac{60-(10+k)}{\sqrt{148}}\right) = 0.125$

	$\frac{50-k}{\sqrt{148}}=1.15035$ $k=50-1.15035\sqrt{148}=36.005\approx 36.0$
(ii)	
(iii)	$X=D+M\sim \mathrm{N}(46,\,148)$ $Y=D+G\sim \mathrm{N}(55,\,104)$ Then $Y-X\sim \mathrm{N}(9,\,252)$ $\mathrm{P}(\lvert Y-X\rvert\le 5)=\mathrm{P}(-5\le Y-X\le 5)$ $=0.21162\approx 0.212$
(iv)	P(started revision late) $=0.8\,\mathrm{P}(D+M>60)+0.2\,\mathrm{P}(D+G>60)$ $=0.16232$ P(played online games | started revision late) $=\frac{0.2\,\mathrm{P}(D+G>60)}{0.16232}$
	$=0.38438\approx 0.384$
(v)	D, M and G are independent normal variables.